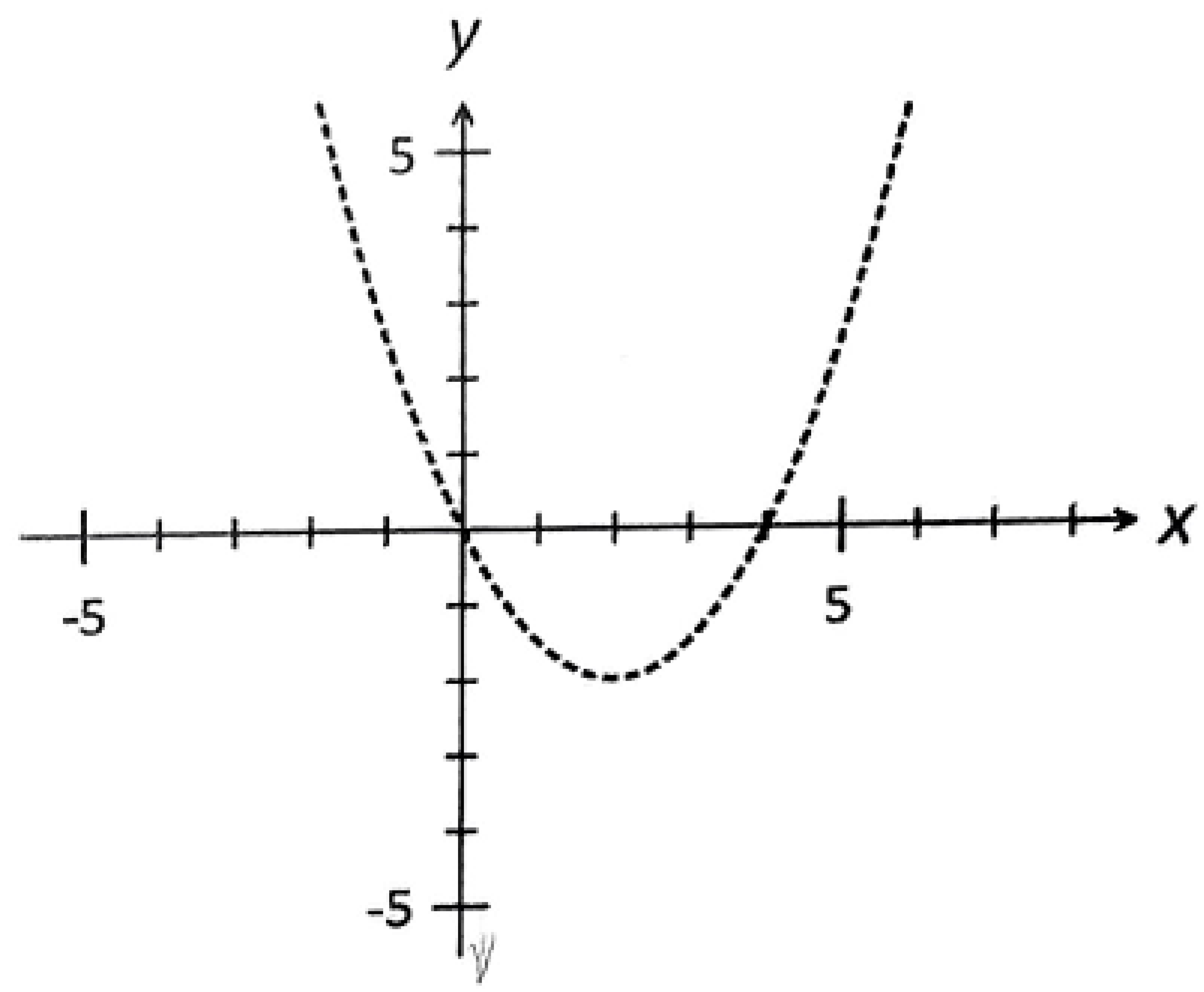
06 Functions II

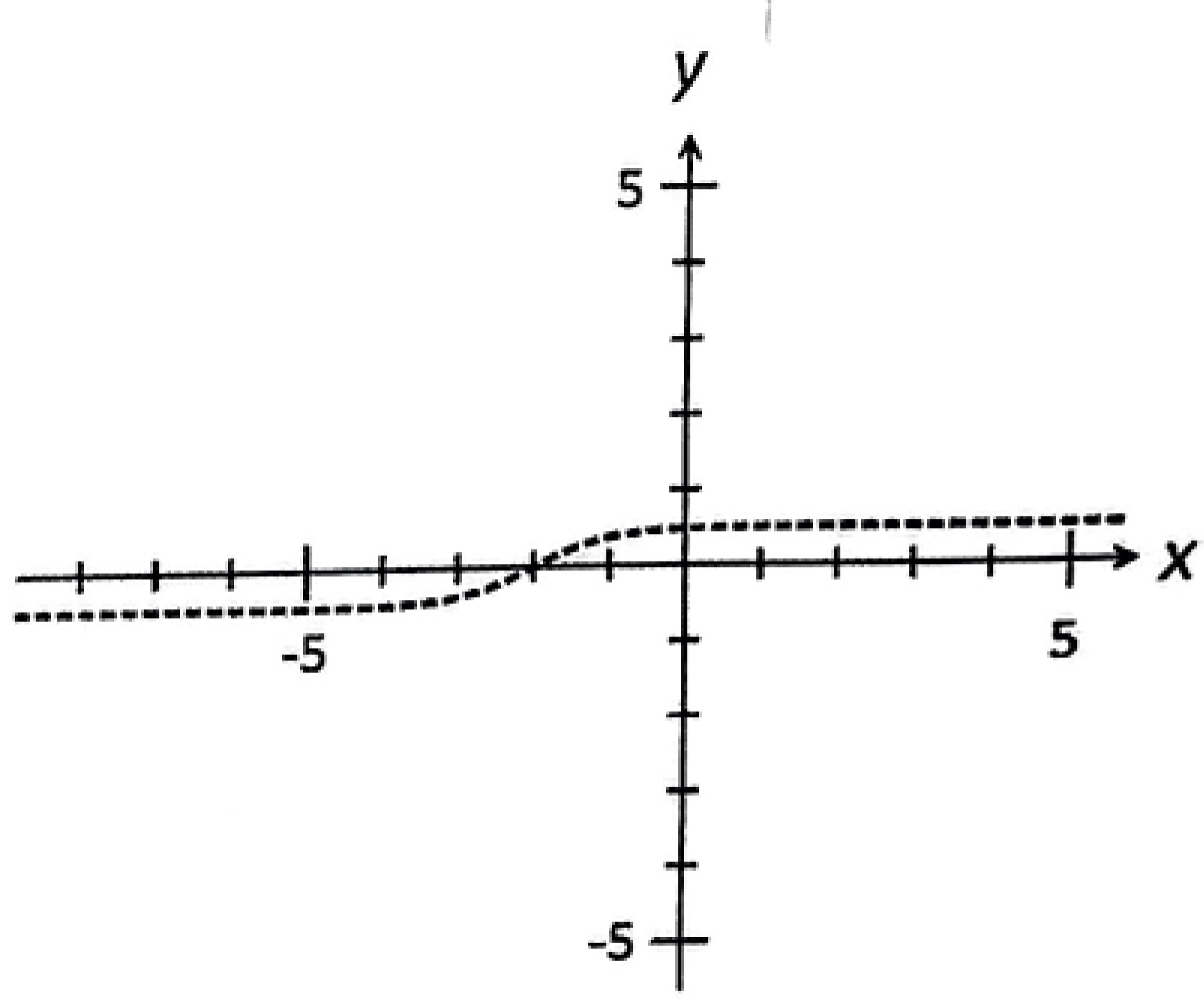
Calculator Free

1. [12 marks: 4, 4, 4]

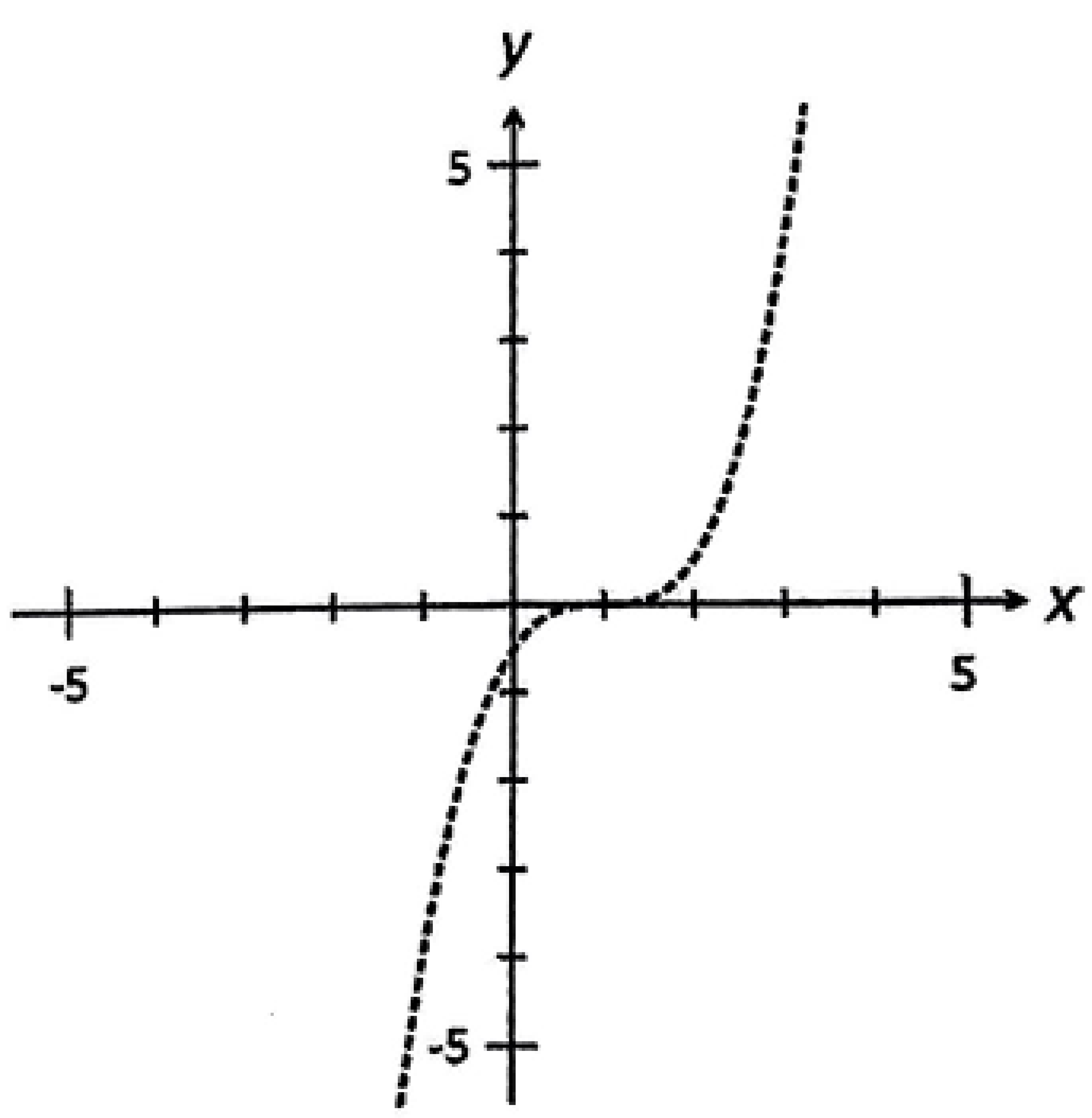
(a) The sketch of $y = f(x)$ is given in the accompanying diagram. Sketch on the same axes the graph of $y = \frac{1}{f(x)}$.



(b) The sketch of $y = \frac{1}{f(x)}$ is given in the accompanying diagram. Sketch on the same axes the graph of $y = f(x)$.



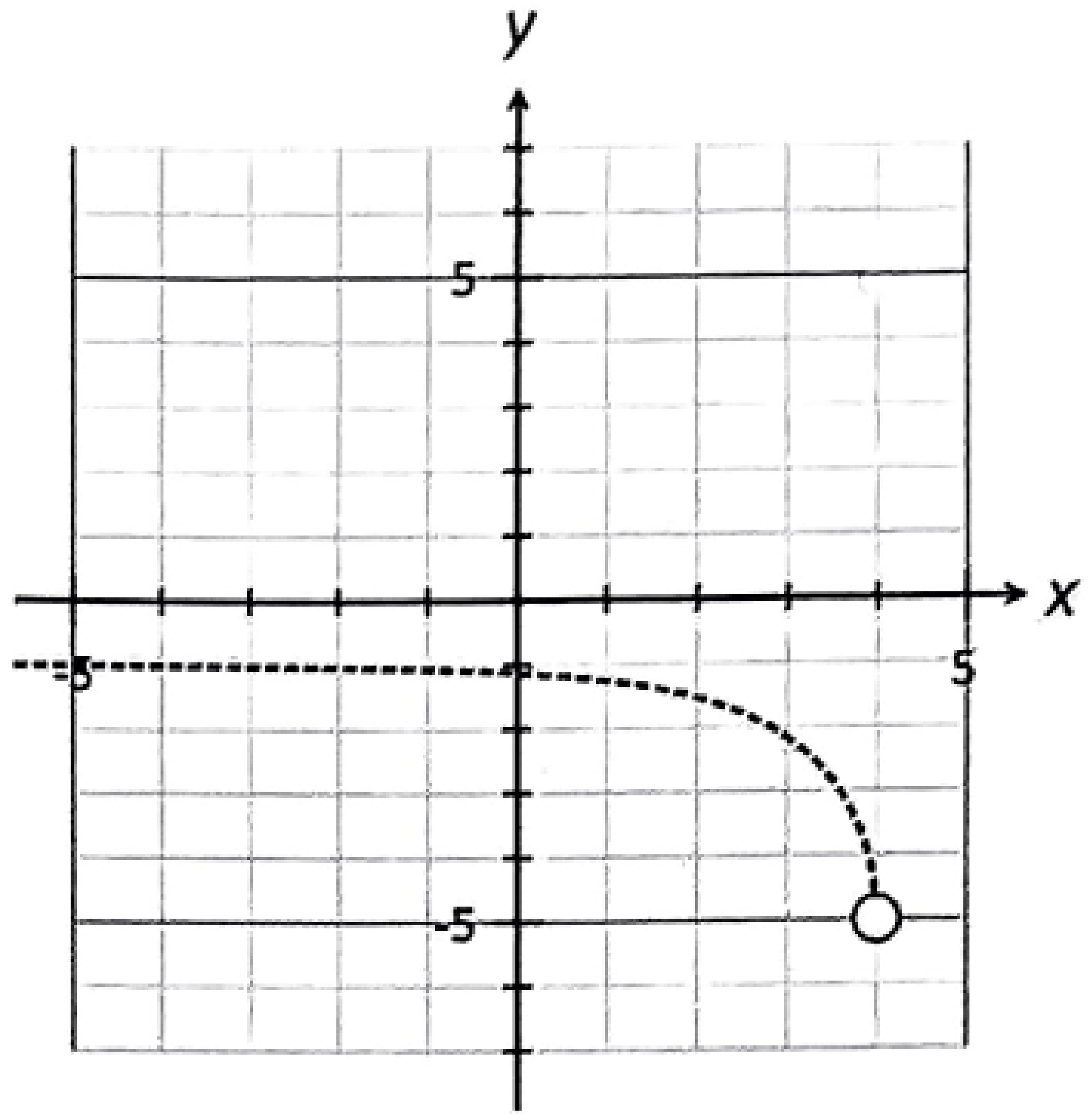
(c) The sketch of $y = f(x)$ is given in the accompanying diagram. Sketch on the same axes the graph of $y^2 = f(x)$.



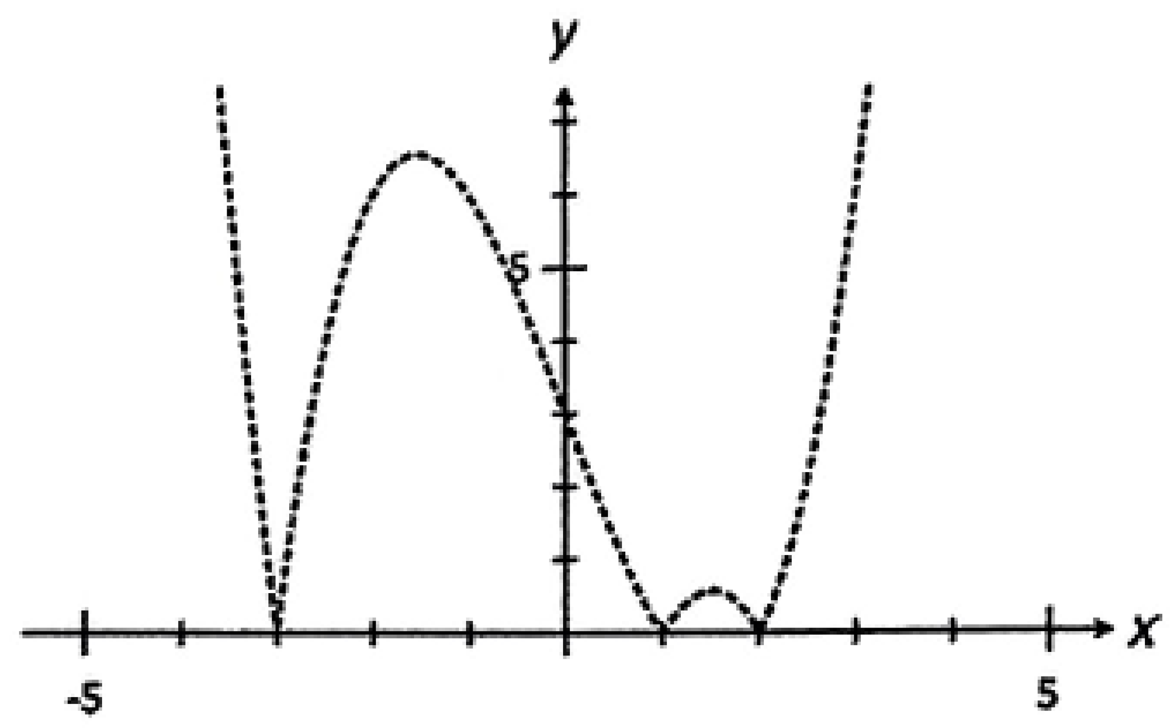
Calculator Free

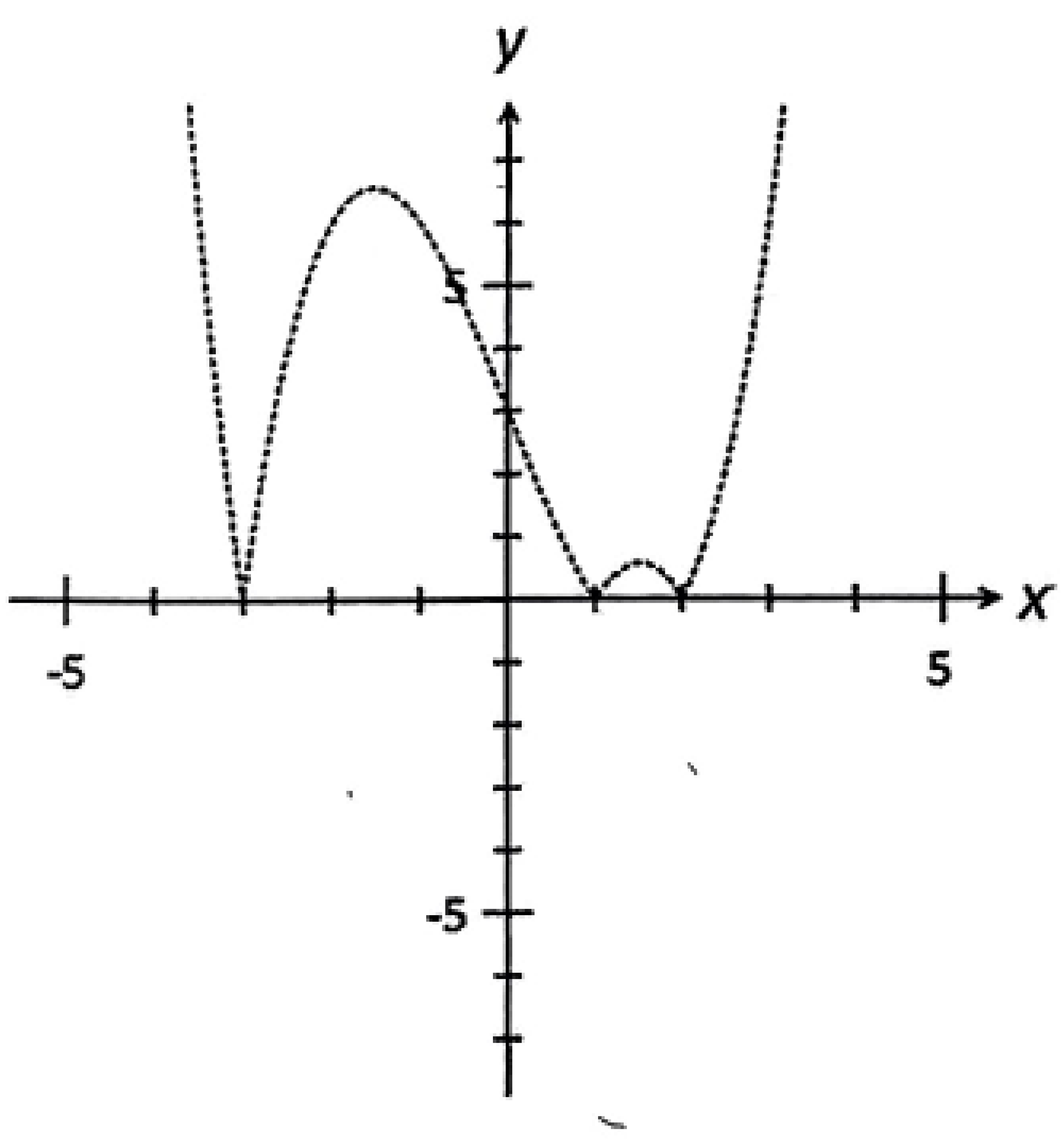
2. [8 marks: 4, 4]

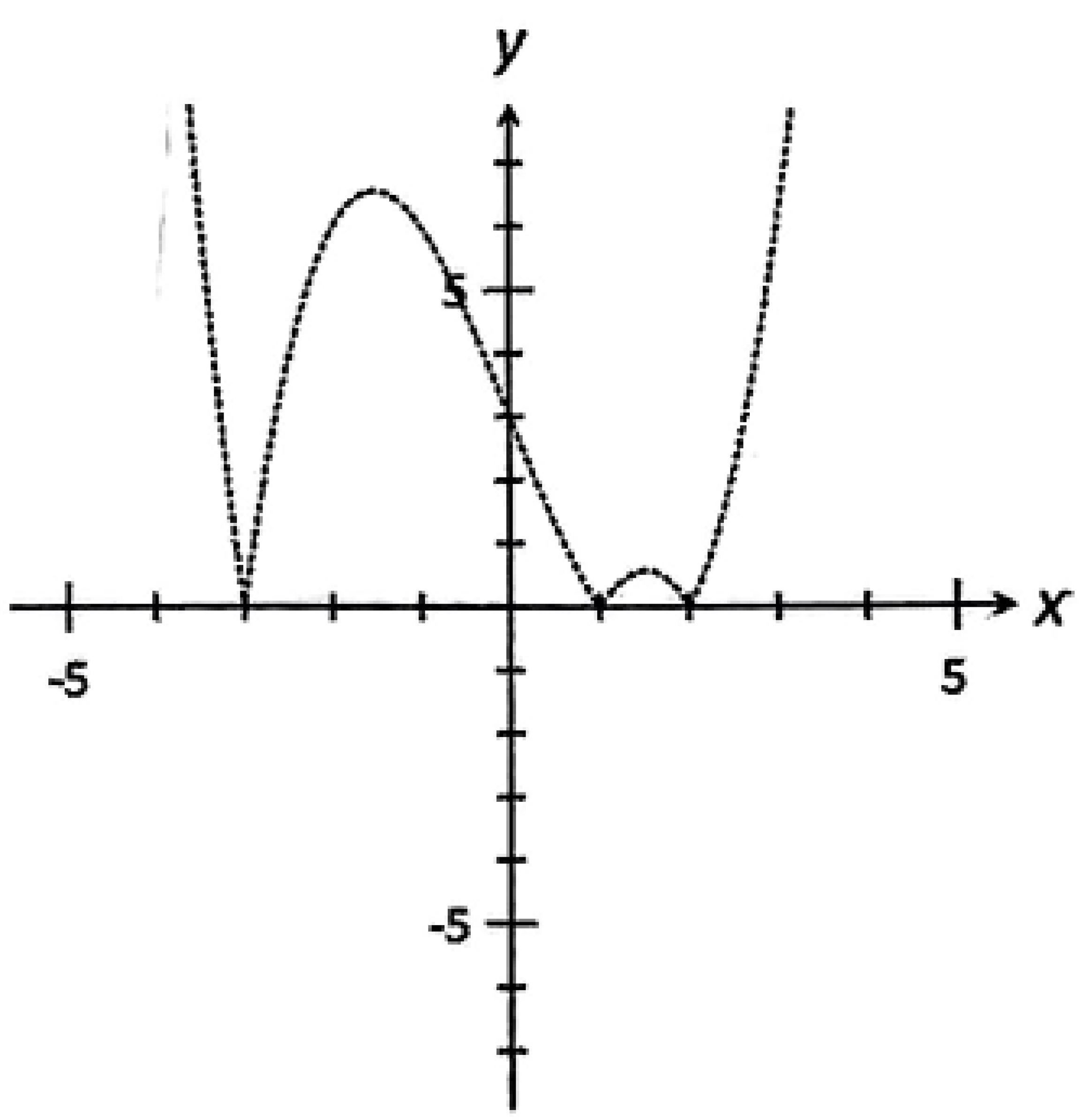
(a) The sketch of $y = f(x)$ is given in the accompanying diagram. Sketch on the same axes the graph of $y = |f(x)|$.



(b) The sketch of $y = |f(x)|$ is given in the accompanying diagram. Sketch on the axes provided below the two possible graphs of $y = f(x)$.



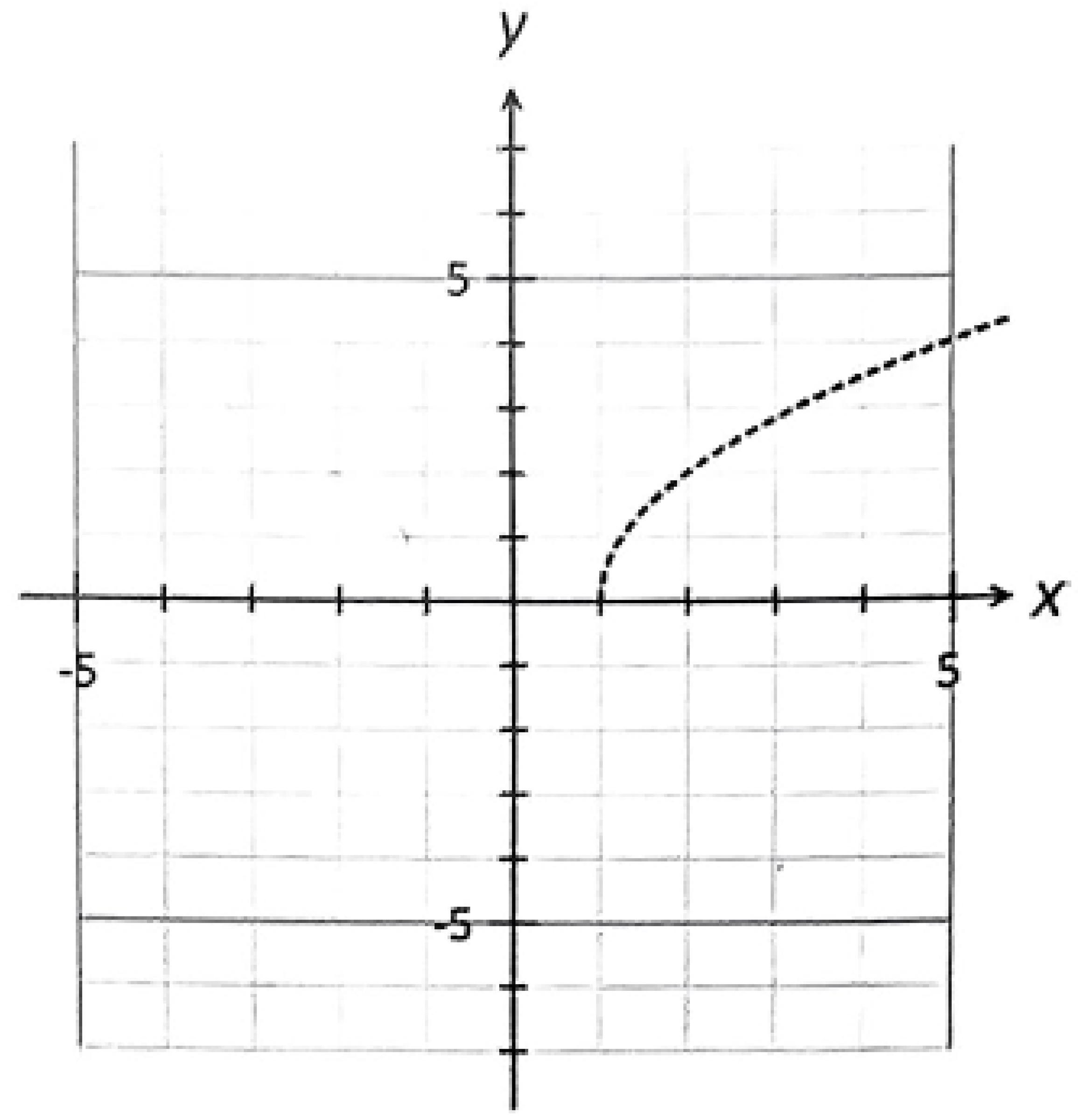




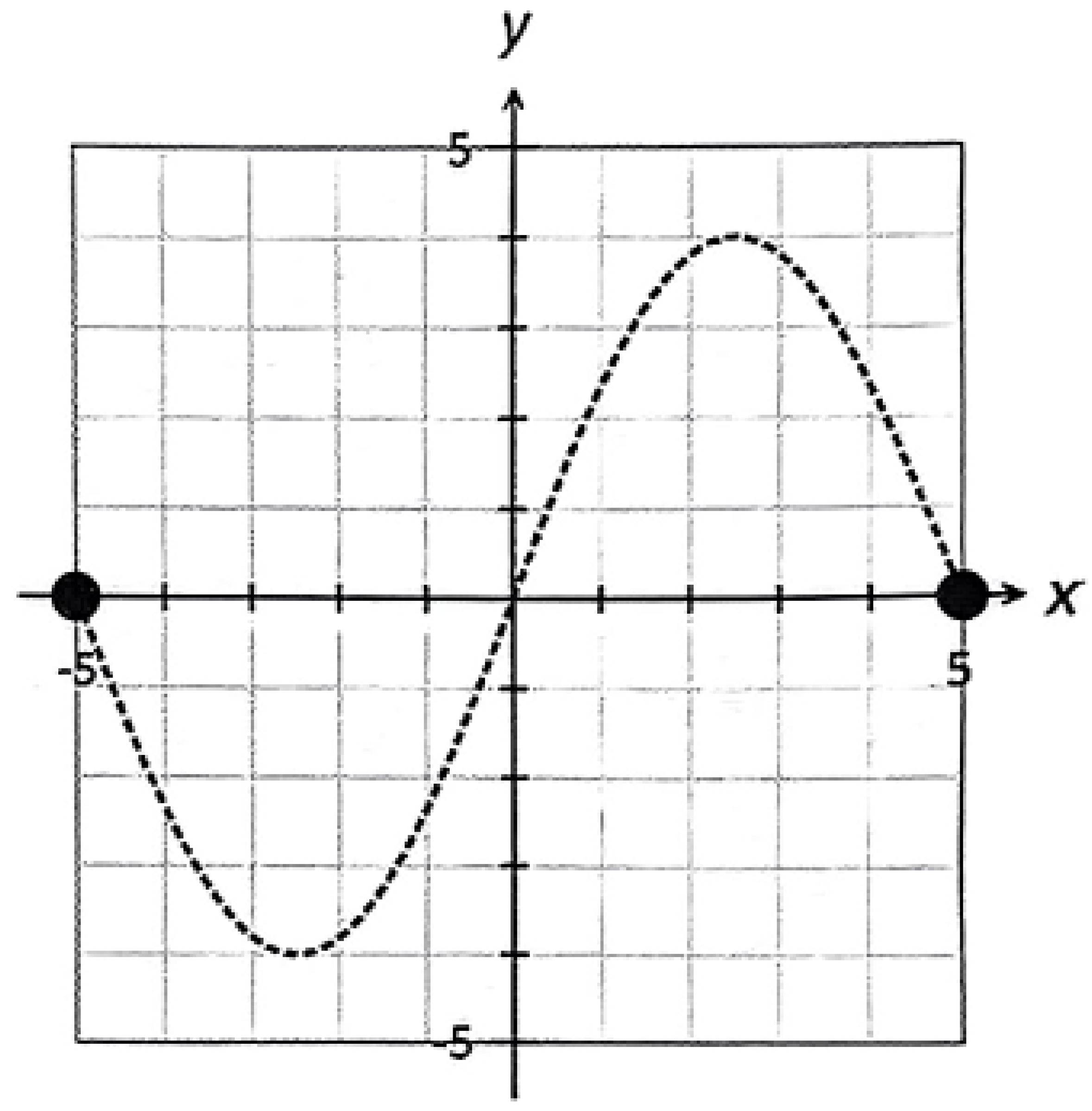
Calculator Free

3. [9 marks: 3, 3, 3]

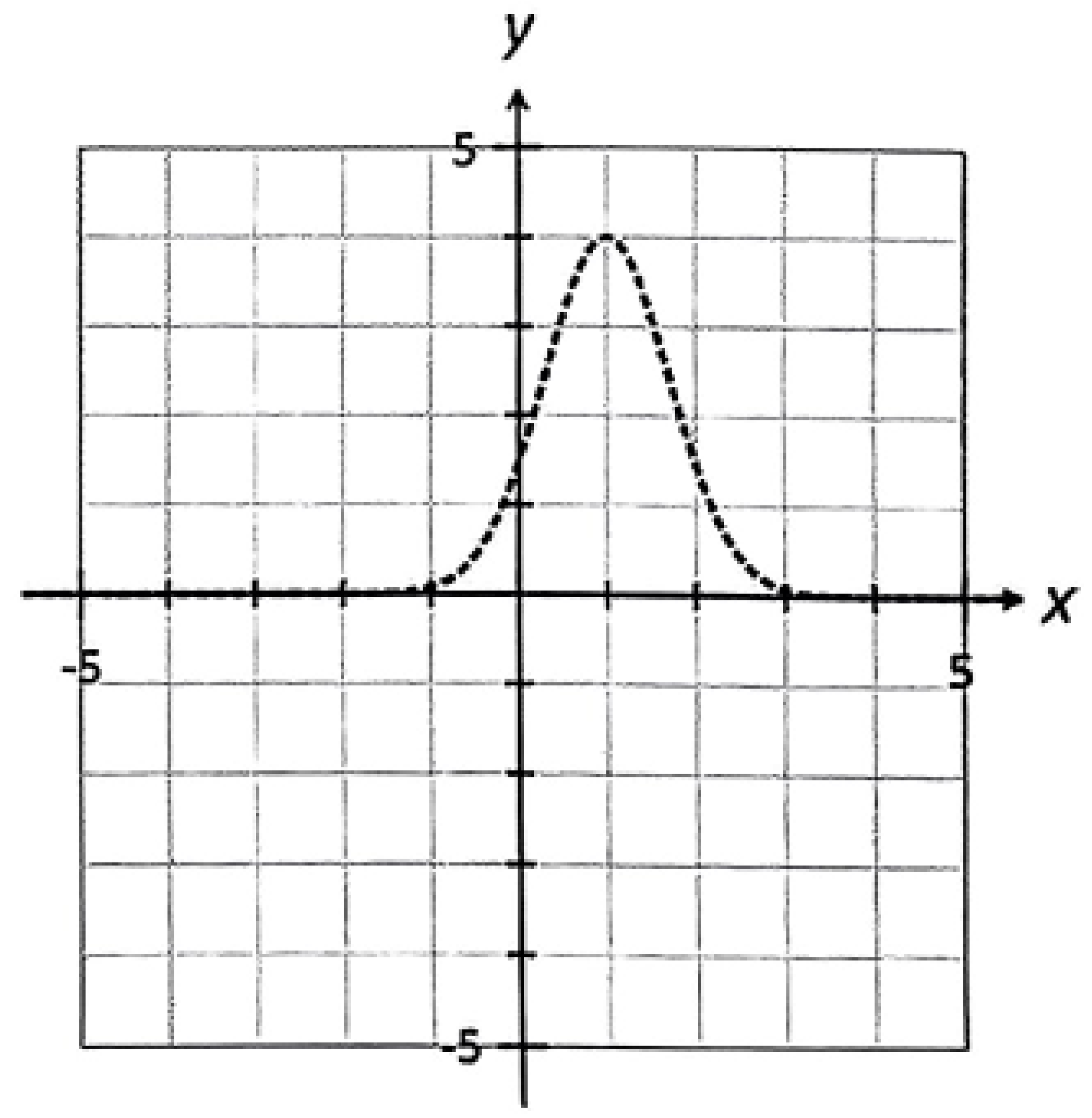
(a) The sketch of $y = f(x)$ is given in the accompanying diagram. Sketch on the same axes the graph of $y = f(|x|)$.



(b) The sketch of $y = f(x)$ is given in the accompanying diagram. Sketch on the same axes the graph of $|y| = f(x)$.



(c) The sketch of $y = f(x)$ is given in the accompanying diagram. Sketch on the same axes the graph of $|y| = f(|x|)$.



Calculator Free

4. [16 marks: 4, 4, 4, 4]

The graph of $y = f(x)$ has intercepts at (2, 0) and (0, –2) and asymptotes with equations $x = 1$ and $y = -1$.

(a) State where they exist, the coordinates of the intercepts and the equations of the asymptotes of the graph of $y = \dfrac{1}{f(x)}$.

(b) State where they exist, the coordinates of the intercepts and the equations of the asymptotes of the graph of $y = |f(x)|$.

(c) State where they exist, the coordinates of the intercepts and the equations of the asymptotes of the graph of $y^2 = f(x)$.

(d) State where they exist, the coordinates of the intercepts and the equations of the asymptotes of the graph of $y = f(|x|)$.

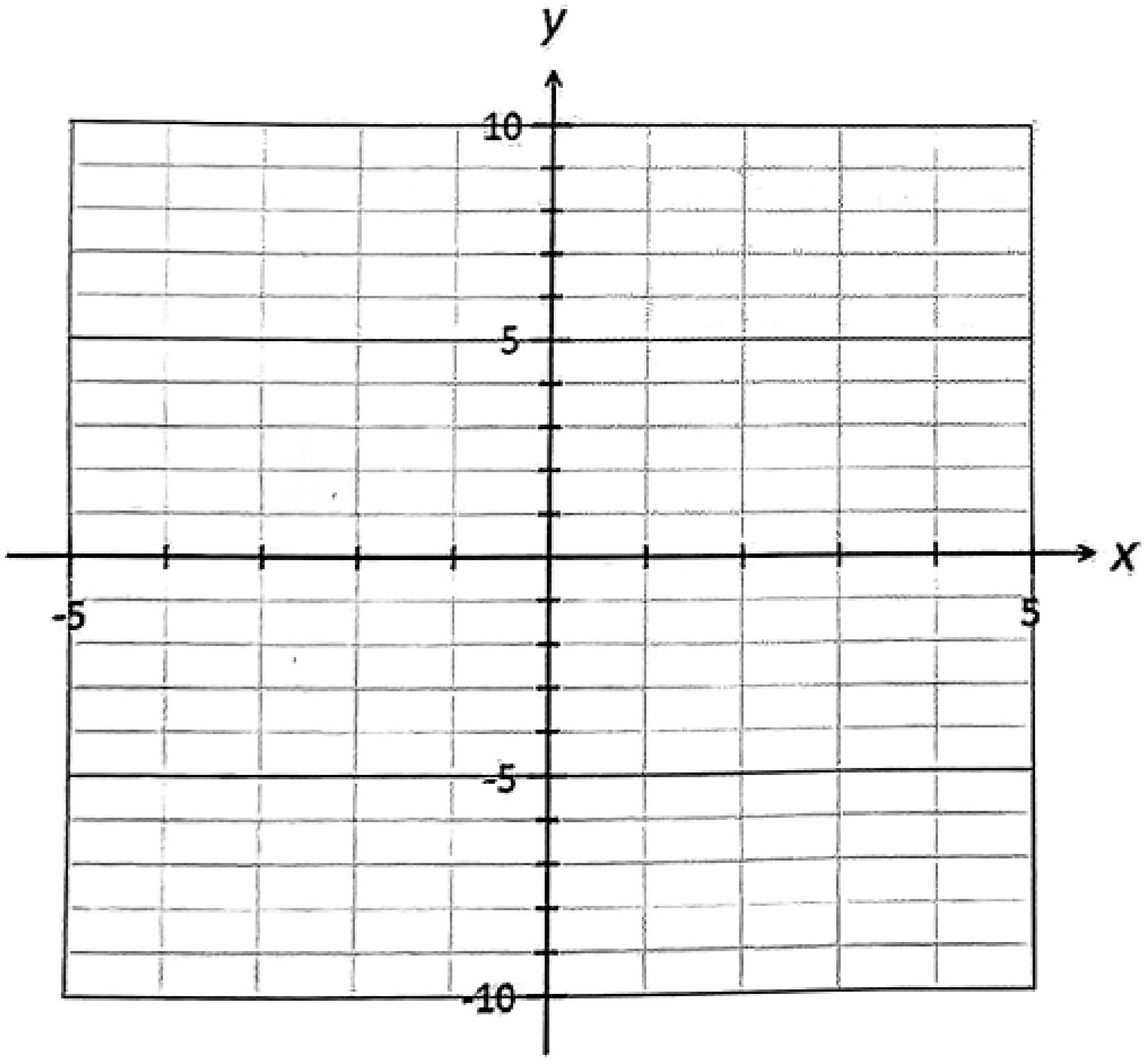
Calculator Free

5. [12 marks: 3, 3, 2, 1, 3] [TISC]

Let $f(x) = |x - 3| - |2x + 1|$.

(a) Rewrite $f(x)$ in piecewise defined form.

(b) On the axes provided below, sketch $y = |x - 3| - |2x + 1|$



Calculator Assumed

5. (c) Determine with reasons if the inverse of f is a function.

(d) If $f^{-1}(x)$ exists only if $x \geq k$. Find the minimum value for k.

(e) For $x \geq k$, find the rule for $f^{-1}(x)$.
Give your answer in piecewise defined form.

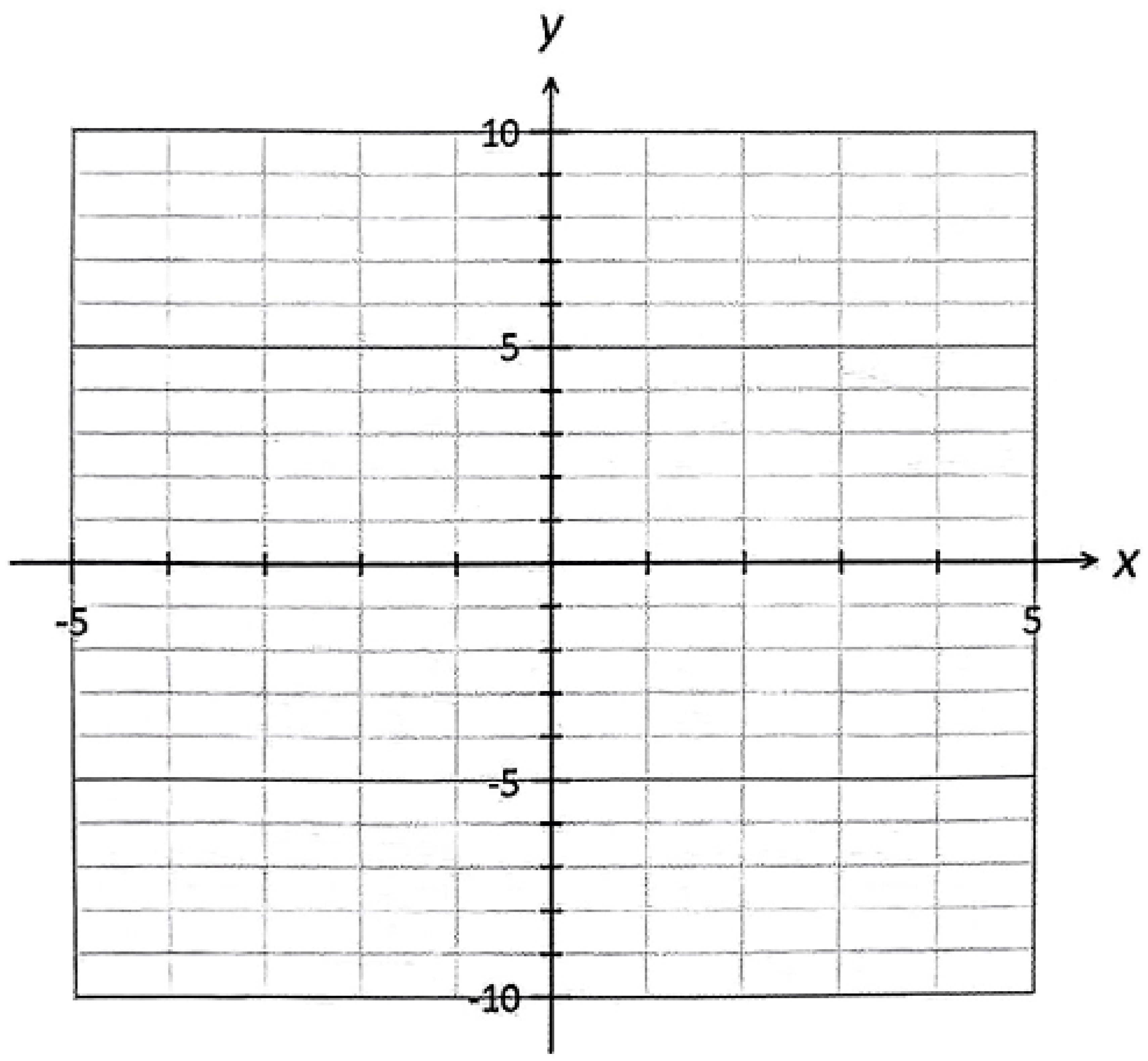
Calculator Free

6. [7 marks: 3, 3, 1] [TISC]

Let $f(x) = x^2 - 3|x| - 4$.

(a) Rewrite $f(x)$ in piecewise defined form.

(b) In the axes provided below, sketch the graph of $y = x^2 - 3|x| - 4$.



(c) Use your sketch above to explain why $f(x)$ does not have an inverse function.

Calculator Free

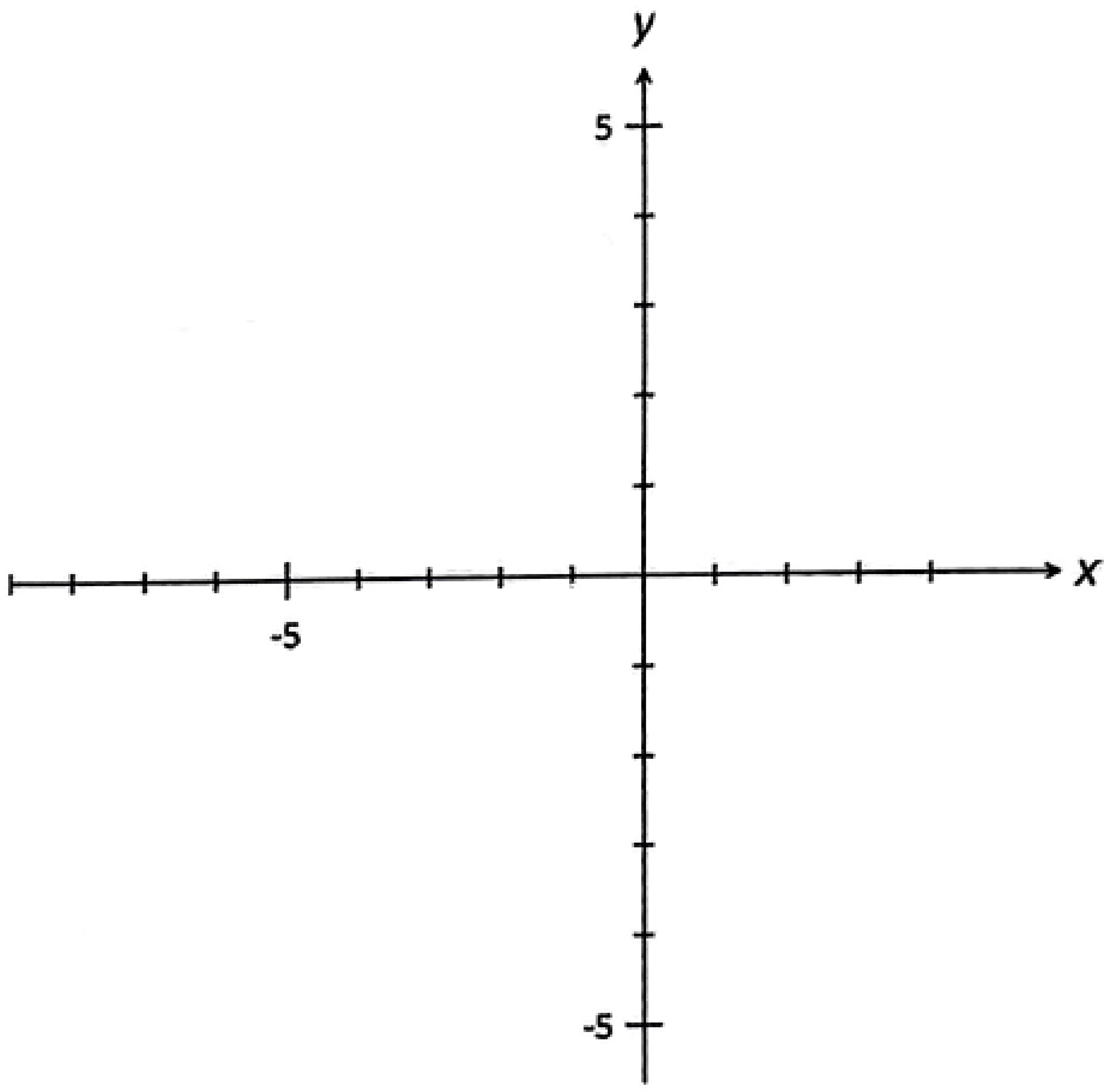
7. [11 marks: 2, 3, 6]

Consider the curve with equation $y = \dfrac{x+2}{x^2-1}$.

(a) State the equation of all asymptotes.

(b) Show that for $x < -2$, $y < 0$.

(c) Sketch this curve. Indicate all intercepts and asymptotes.



Calculator Free

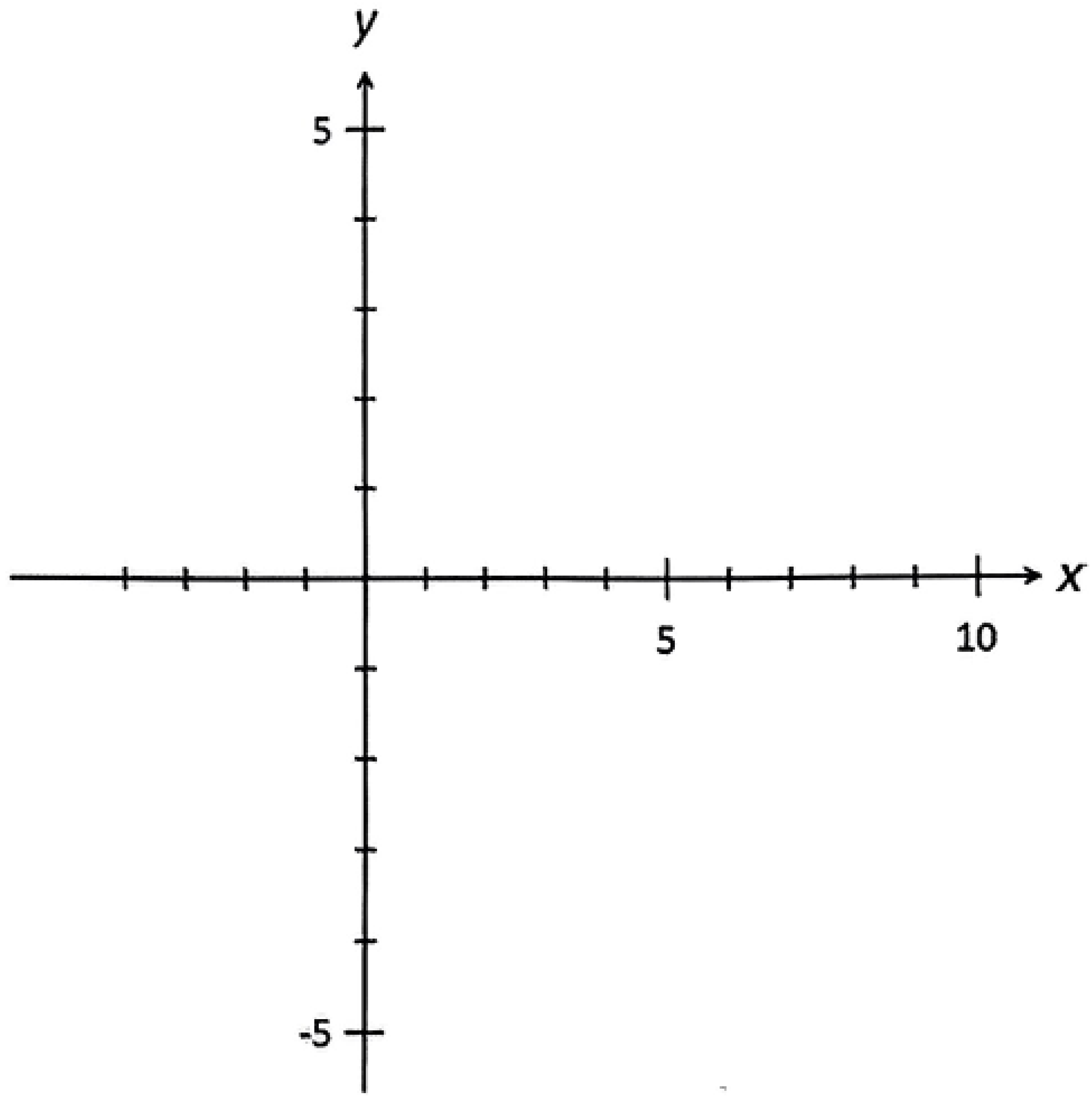
8. [9 marks: 2, 3, 4]

Consider the curve with equation $y = \dfrac{x^2+x-2}{x^2-2x-8}$.

(a) State the equation of all asymptotes.

(b) Identify the point of discontinuity on this curve.

(c) Sketch this curve on the axes provided below.



Calculator Free

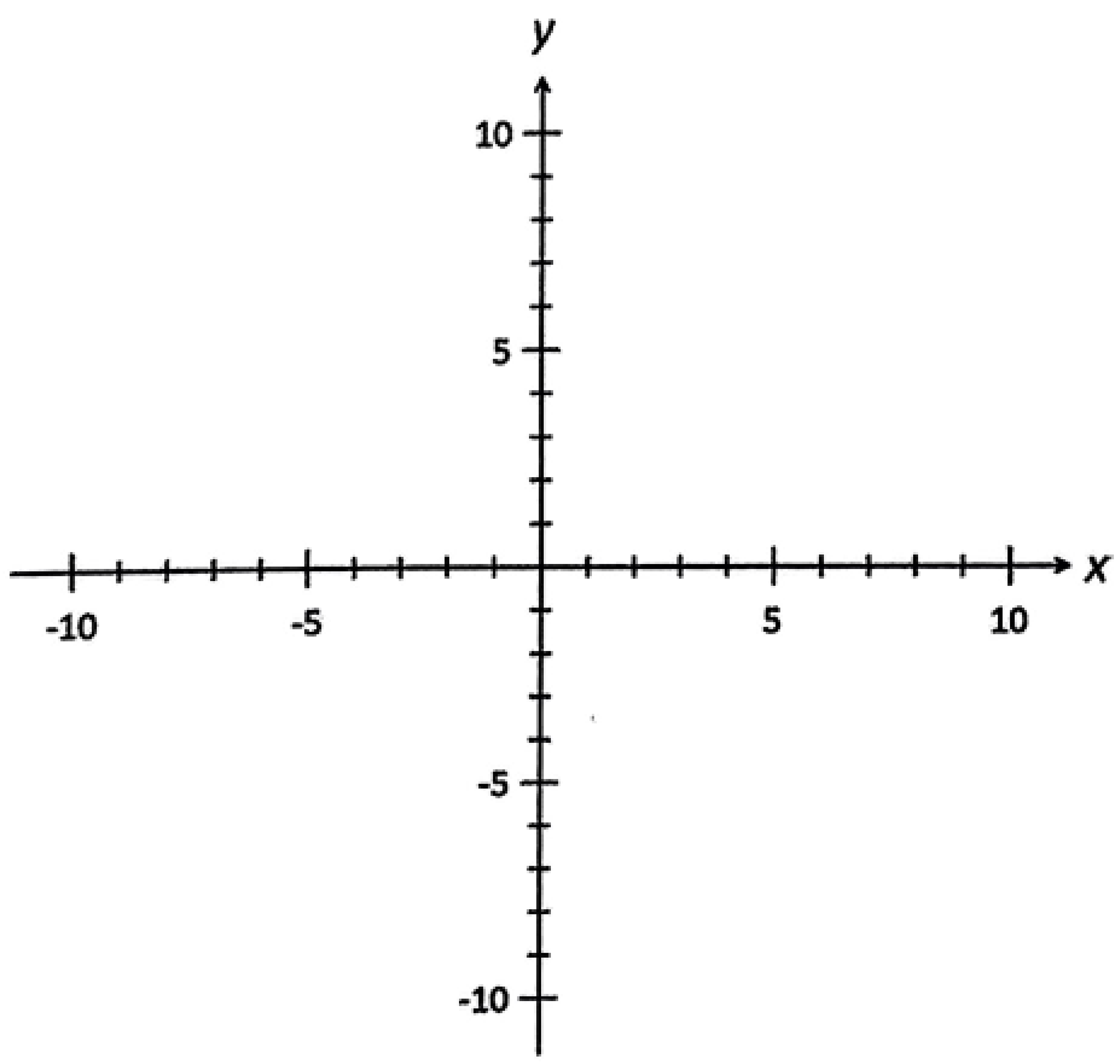
9. [11 marks: 4, 2, 5]

Consider the curve with equation $y = \dfrac{x^2 + x - 6}{x - 1}$.

(a) Rewrite the equation of the curve in the form $y \equiv \dfrac{P(x)}{Q(x)} + ax + b$ where $\dfrac{P(x)}{Q(x)}$ is a rational proper fraction and a and b are real constants.

(b) State the equations of all asymptotes of this curve.

(c) On the axes provided below sketch the graph of $y = \dfrac{x^2 + x - 6}{x - 1}$.
Indicate all intercepts and asymptotes.



Calculator Free

10. [10 marks: 3, 4, 3]

Let $y = \dfrac{ax^3 + bx + c}{x^2 + k}$ where a, b, c and k are real constants.

(a) Rewrite $y = \dfrac{ax^3 + bx + c}{x^2 + k}$ in the form $y = \dfrac{P(x)}{Q(x)} + px + q$ where $\dfrac{P(x)}{Q(x)}$ is a rational proper fraction and p and q are real constants.

(b) The curve has intercepts only at $(-2, 0)$ and $(0, -1)$ and asymptotes with equation $y = x$.

(i) Determine the value of a and express b and c in terms of k.

(ii) Give a possible set of values for b, c and k if in addition, the curve has no singularities and no vertical asymptotes.

Calculator Free

19. (b) For $f(x) = \left|\dfrac{2x-1}{x-3}\right|$ where $\frac{1}{2} \le x < 3$, find the rule for f^{-1}.

For $\frac{1}{2} \le x < 3$: $y = f(x) = -\left(\dfrac{2x-1}{x-3}\right)$ ✓

$xy - 3y = 1 - 2x$ ✓

$x = \dfrac{1+3y}{2+y}$

Hence, $f^{-1}(x) = \dfrac{1+3x}{2+x}$ ✓

---

20. [6 marks: 3, 3]

Consider $f(x) = \sin 2x$ and $g(x) = \cos \dfrac{x}{2}$.

(a) $f(x)$ is a one-to-one function within the domain $-a \le x \le a$.
Determine the largest possible value for $|a|$.
Hence, determine the rule for $f^{-1}(x)$ and state the corresponding range.

Max value for $|a| = \frac{\pi}{4}$ ✓

$y = \sin 2x \Rightarrow x = \frac{1}{2}\sin^{-1} y$

Hence, $f^{-1}(x) = \frac{1}{2}\sin^{-1} x$. ✓

Range: $[-\frac{\pi}{4}, \frac{\pi}{4}]$ ✓

(b) $g(x)$ is a one-to-one function within the domain $0 \le x \le b$.
Determine the largest possible value for b.
Hence, determine the rule for $g^{-1}(x)$ and state the corresponding range.

Max value for $b = 2\pi$ ✓

$y = \cos \frac{x}{2} \Rightarrow x = 2\cos^{-1} y$

Hence, $f^{-1}(x) = 2\cos^{-1} x$. ✓

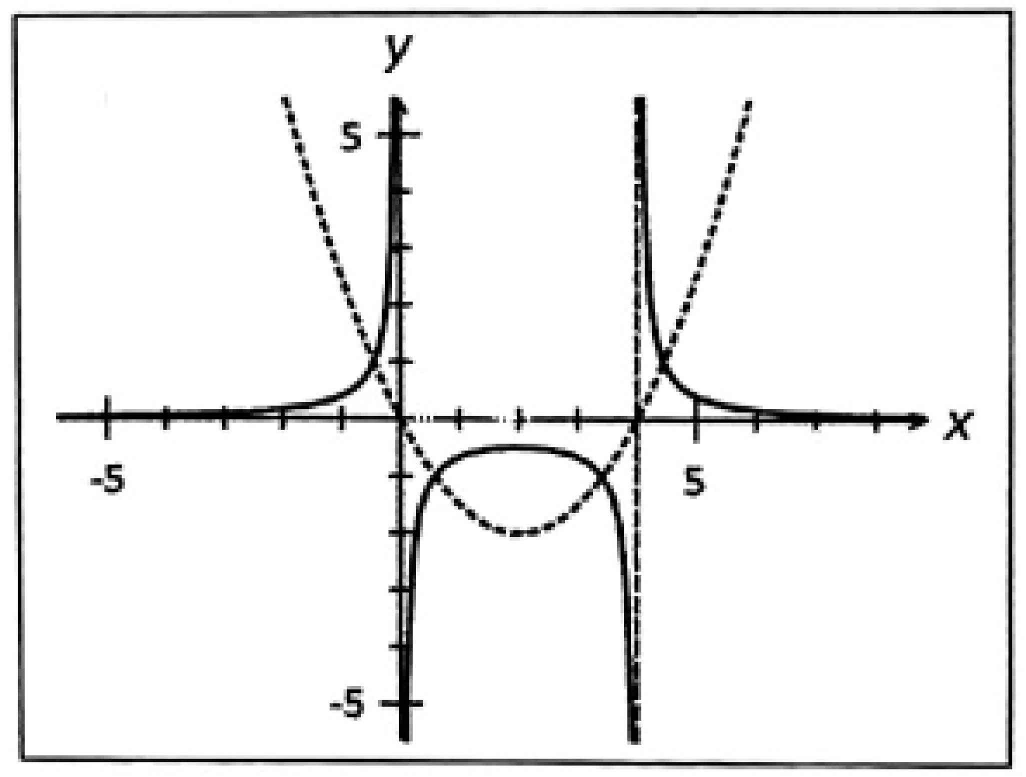
Range: $[0, 2\pi]$ ✓

06 Functions II

Calculator Free

1. [12 marks: 4, 4, 4]

(a) The sketch of $y = f(x)$ is given in the accompanying diagram. Sketch on the same axes the graph of $y = \dfrac{1}{f(x)}$.

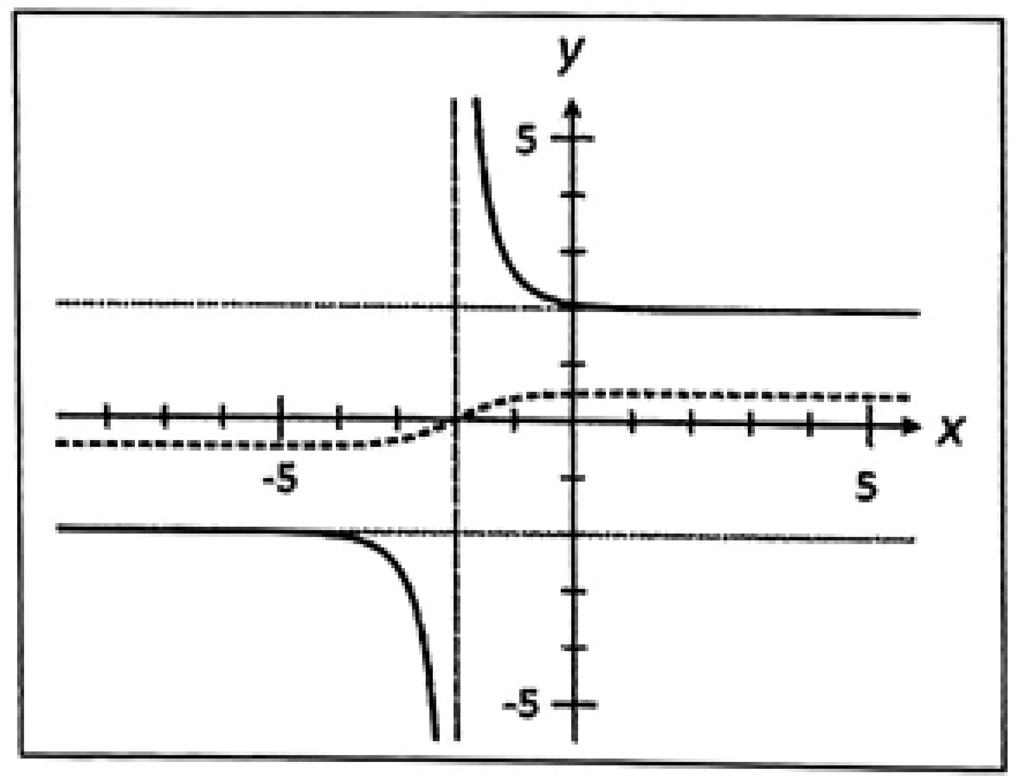


Asymptotes: $y = 0, x = 0, x = 4$ ✓✓

Max point $(2, \frac{1}{2})$ ✓

All correct ✓

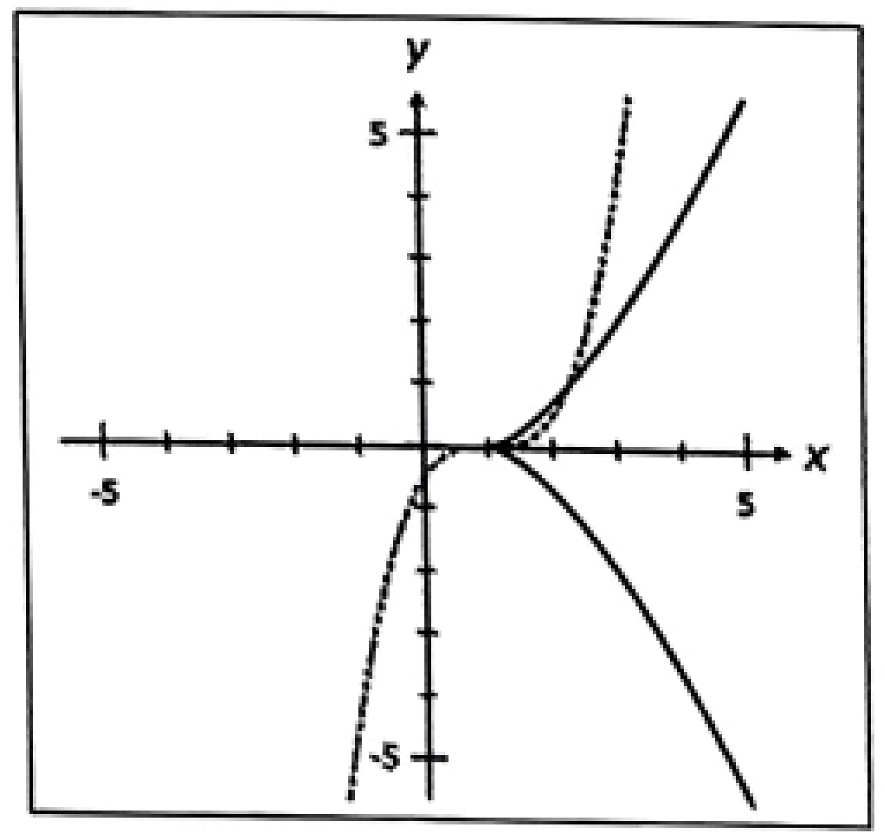
(b) The sketch of $y = \dfrac{1}{f(x)}$ is given in the accompanying diagram. Sketch on the same axes the graph of $y = f(x)$.



Asymptotes: $y = -2, y = 2, x = -2$ ✓✓✓

All correct ✓

(c) The sketch of $y = f(x)$ is given in the accompanying diagram. Sketch on the same axes the graph of $y^2 = f(x)$.



Domain: $[1, \infty)$ ✓

Symmetrical about the x-axis. ✓

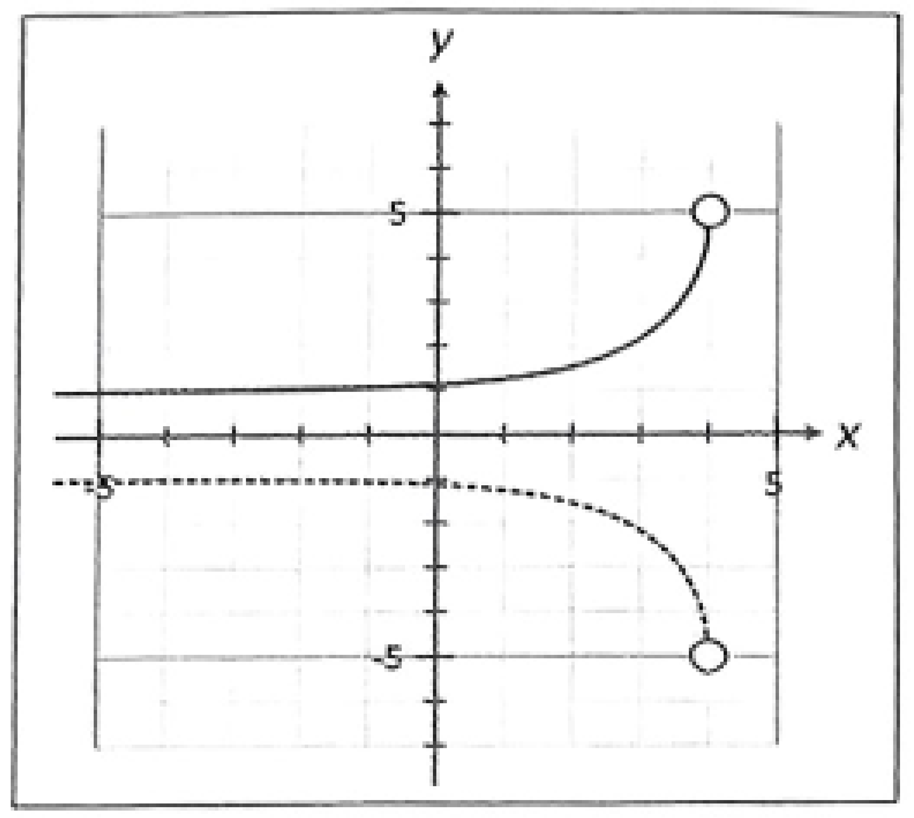
Both graphs intersect when $y = 1$ ✓

All correct. ✓

Calculator Free

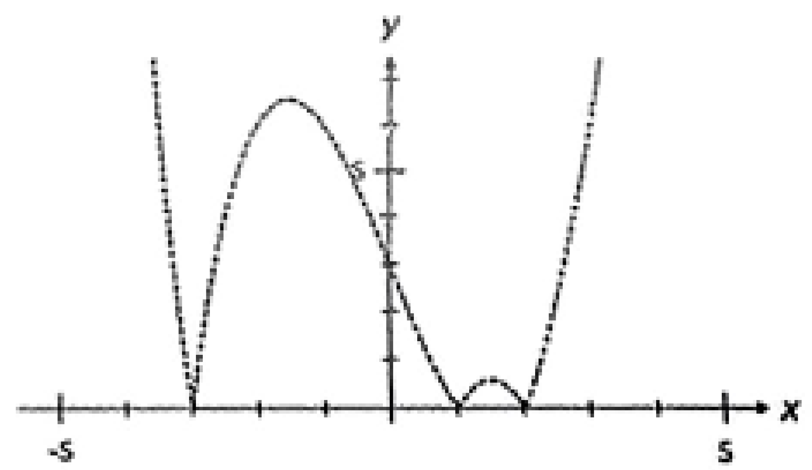
2. [8 marks: 4, 4]

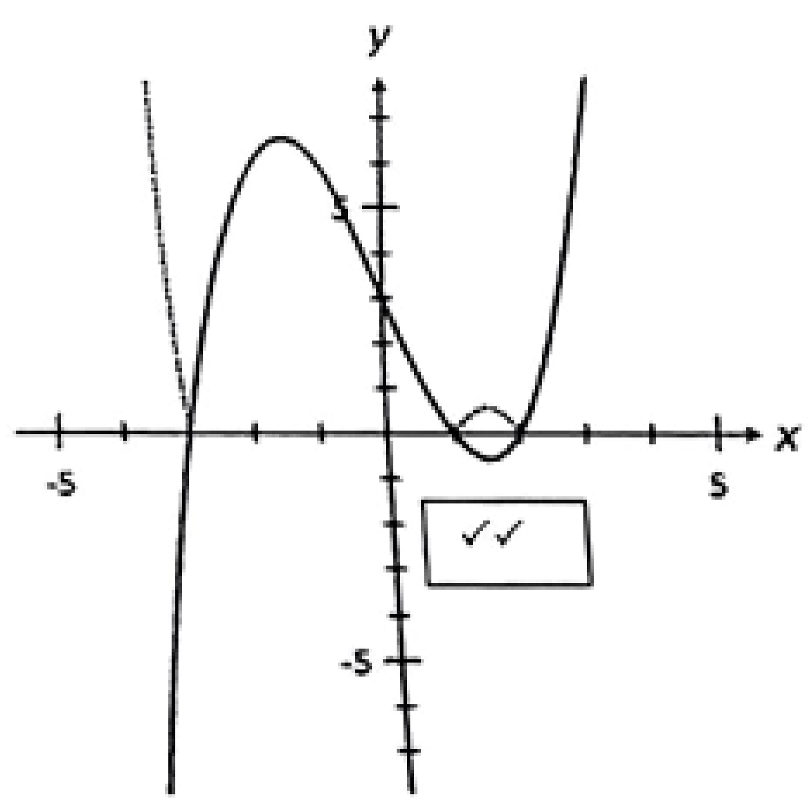
(a) The sketch of $y = f(x)$ is given in the accompanying diagram. Sketch on the same axes the graph of $y = |f(x)|$.

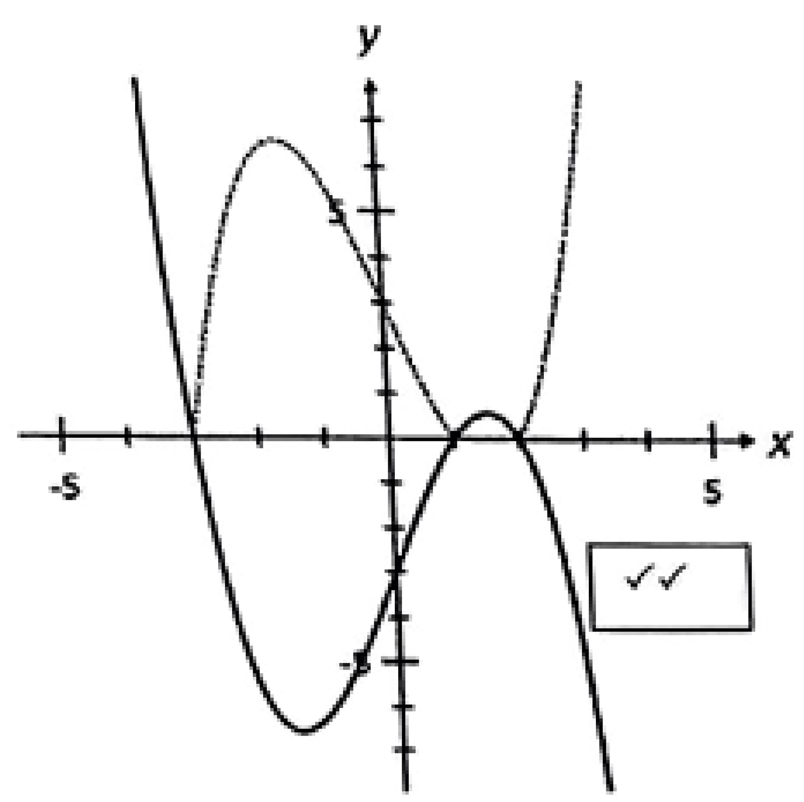


Reflected about x-axis	✓
End point: (5, 5)	✓
Asymptote: $y = 1$	✓
All correct.	✓

(b) The sketch of $y = |f(x)|$ is given in the accompanying diagram. Sketch on the axes provided below the two possible graphs of $y = f(x)$.



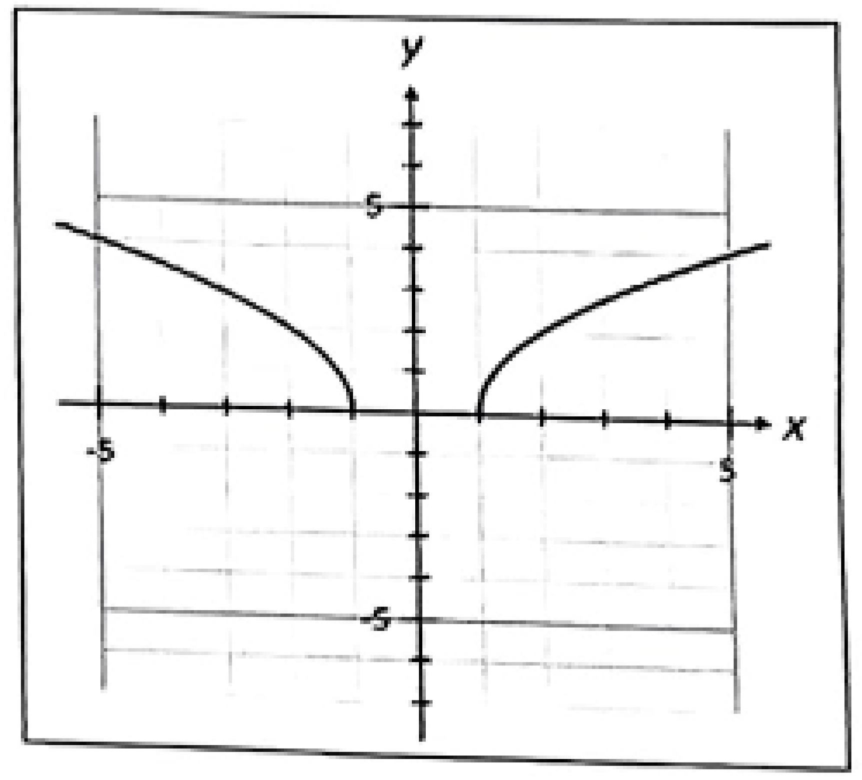




Calculator Free

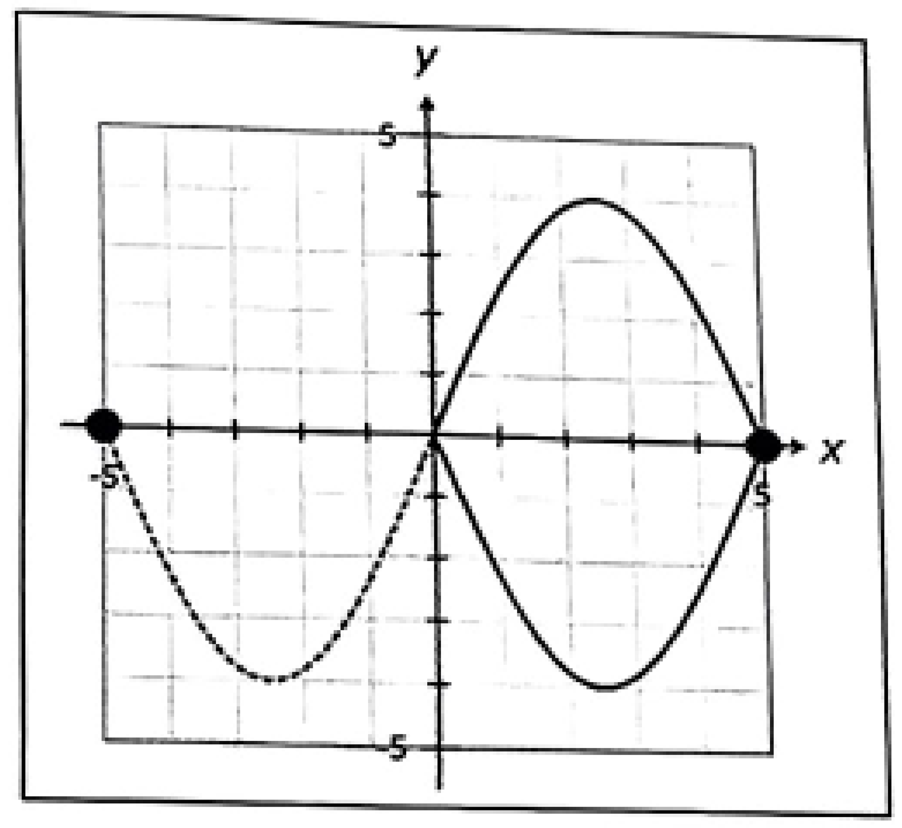
3. [9 marks: 3, 3, 3]

(a) The sketch of $y = f(x)$ is given in the accompanying diagram. Sketch on the same axes the graph of $y = f(|x|)$.



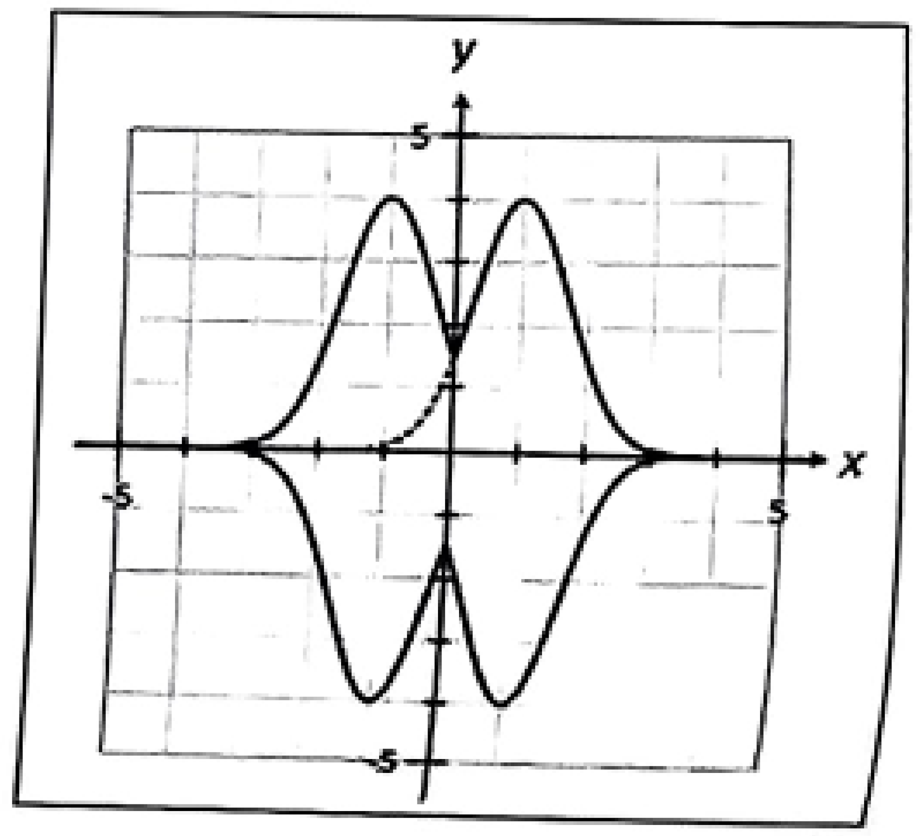
Symmetrical about y-axis	✓
Domain: $(-\infty,-1] \cup [1,\infty)$	✓
All correct	✓

(b) The sketch of $y = f(x)$ is given in the accompanying diagram. Sketch on the same axes the graph of $|y| = f(x)$.



Domain: [0, 5]	✓
Range: [−4, 4]	✓
Symmetrical about x-axis	✓

(c) The sketch of $y = f(x)$ is given in the accompanying diagram. Sketch on the same axes the graph of $|y| = f(|x|)$.



Symmetrical about x-axis	✓
Symmetrical about y-axis	✓
All correct	✓

Calculator Free

4. [16 marks: 4, 4, 4, 4]

The graph of $y = f(x)$ has intercepts at (2, 0) and (0, −2) and asymptotes with equations $x = 1$ and $y = -1$.

(a) State where they exist, the coordinates of the intercepts and the equations of the asymptotes of the graph of $y = \dfrac{1}{f(x)}$.

Intercepts: (1, 0) and $(0, -\frac{1}{2})$. ✓✓
Asymptotes: $x = 2$, $y = -1$ ✓✓

(b) State where they exist, the coordinates of the intercepts and the equations of the asymptotes of the graph of $y = |f(x)|$.

Intercepts: (2, 0), (0, 2) ✓✓
Asymptotes: $x = 1$, $y = 1$ ✓✓

(c) State where they exist, the coordinates of the intercepts and the equations of the asymptotes of the graph of $y^2 = f(x)$.

Intercepts: (2, 0) ✓✓
Asymptotes: $x = 1$ ✓✓

(d) State where they exist, the coordinates of the intercepts and the equations of the asymptotes of the graph of $y = f(|x|)$.

Intercepts: (2, 0), (−2, 0), (0, −2) ✓✓
Asymptotes: $y = -1$, $x = -1$, $x = 1$ ✓✓

Calculator Free

5. [12 marks: 3, 3, 2, 1, 3] [TISC]

Let $f(x) = |x-3| - |2x+1|$.

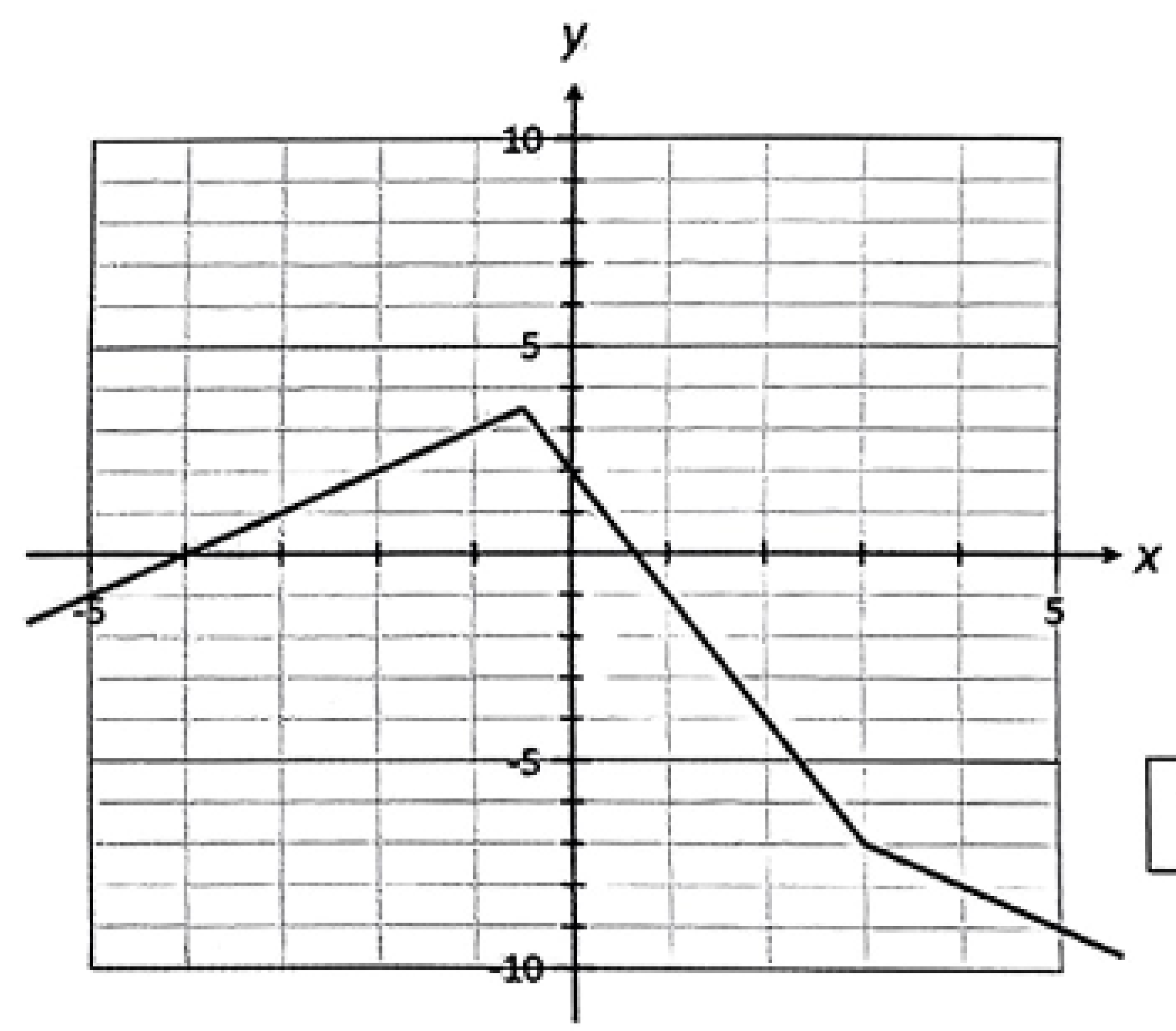
(a) Rewrite $f(x)$ in piecewise defined form.

Critical points: $x = -\frac{1}{2}, 3$ ✓

$$f(x) = \begin{cases} -(x-3) - [-(2x+1)] & x < -\frac{1}{2} \\ -(x-3) - (2x+1) & -\frac{1}{2} \le x \le 3 \\ (x-3) - (2x+1) & x > 3 \end{cases}$$ ✓

$$= \begin{cases} x+4 & x < -\frac{1}{2} \\ -3x+2 & -\frac{1}{2} \le x \le 3 \\ -(x+4) & x > 3 \end{cases}$$ ✓

(b) On the axes provided below, sketch $y = |x-3| - |2x+1|$



✓✓✓

Calculator Assumed

5. (c) Determine with reasons if the inverse of f is a function.

> No, inverse of f is not a function. ✓
> Graph of $f(x)$ fails the horizontal line test. ✓
> OR $f(-2) = f(0) = 2 \Rightarrow f(x)$ is not a one-to-one function.

(d) If $f^{-1}(x)$ exists only if $x \geq k$. Find the minimum value for k.

> $k = -\frac{1}{2}$ ✓

(e) For $x \geq k$, find the rule for $f^{-1}(x)$.
Give your answer in piecewise defined form.

> Critical points: $x = -7, \frac{7}{2}$ ✓
>
> $f^{-1}(x) = \begin{cases} -(4+x) & x < -7 \\ \frac{2-x}{3} & -7 \leq x \leq \frac{7}{2} \end{cases}$ ✓✓

Calculator Free

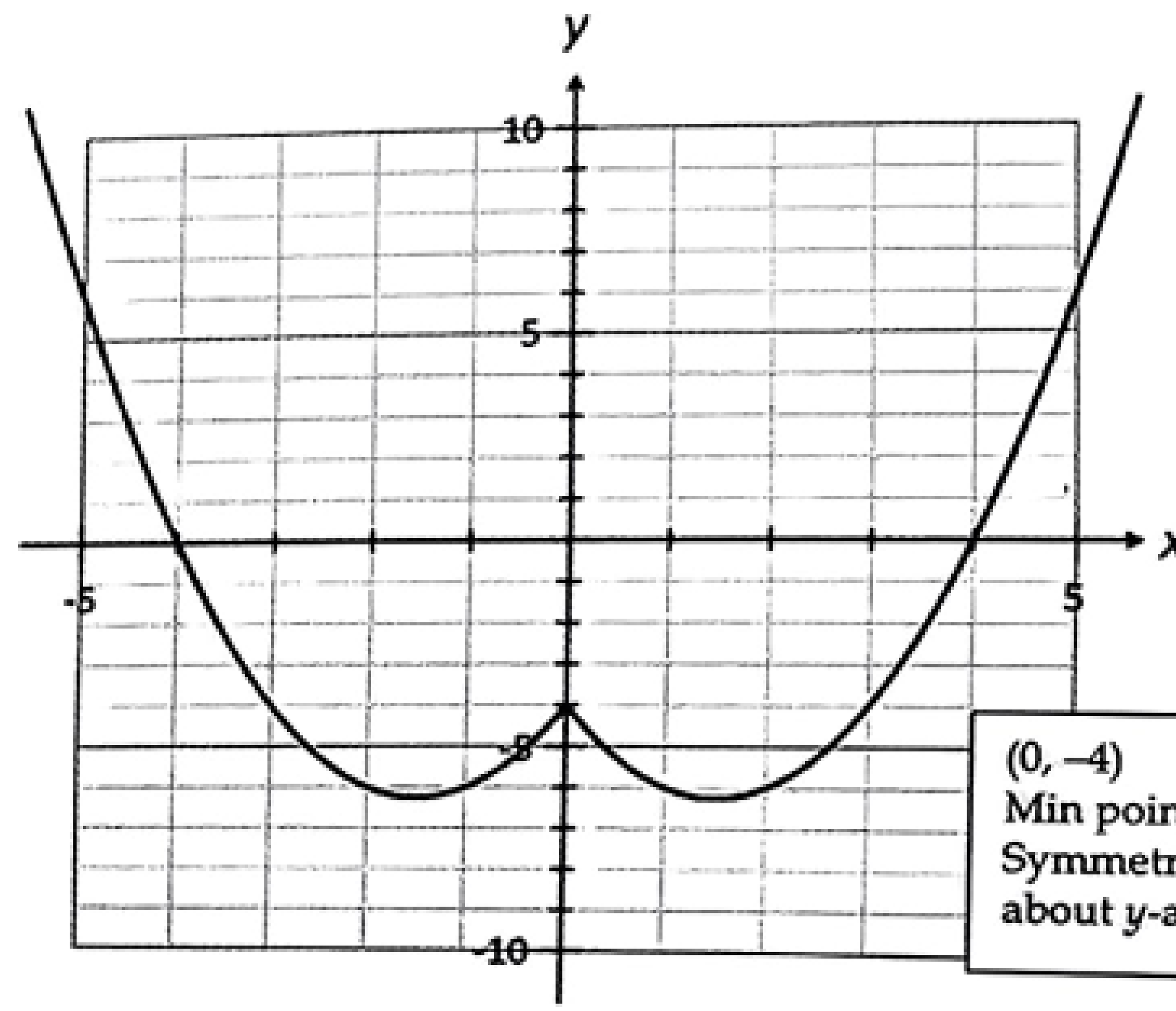
6. [7 marks: 3, 3, 1] [TISC]

Let $f(x) = x^2 - 3|x| - 4$.

(a) Rewrite $f(x)$ in piecewise defined form.

> Critical point: $x = 0$ ✓
>
> $f(x) = \begin{cases} x^2 + 3x - 4 & x < 0 \\ x^2 - 3x - 4 & x \geq 0 \end{cases}$ ✓✓

(b) In the axes provided below, sketch the graph of $y = x^2 - 3|x| - 4$.



> (0, −4) ✓
> Min points ✓
> Symmetrical about y-axis ✓

(c) Use your sketch above to explain why $f(x)$ does not have an inverse function.

> Graph of $f(x)$ fails the horizontal line test. ✓

Calculator Free

7. [11 marks: 2, 3, 6]

Consider the curve with equation $y = \dfrac{x+2}{x^2-1}$.

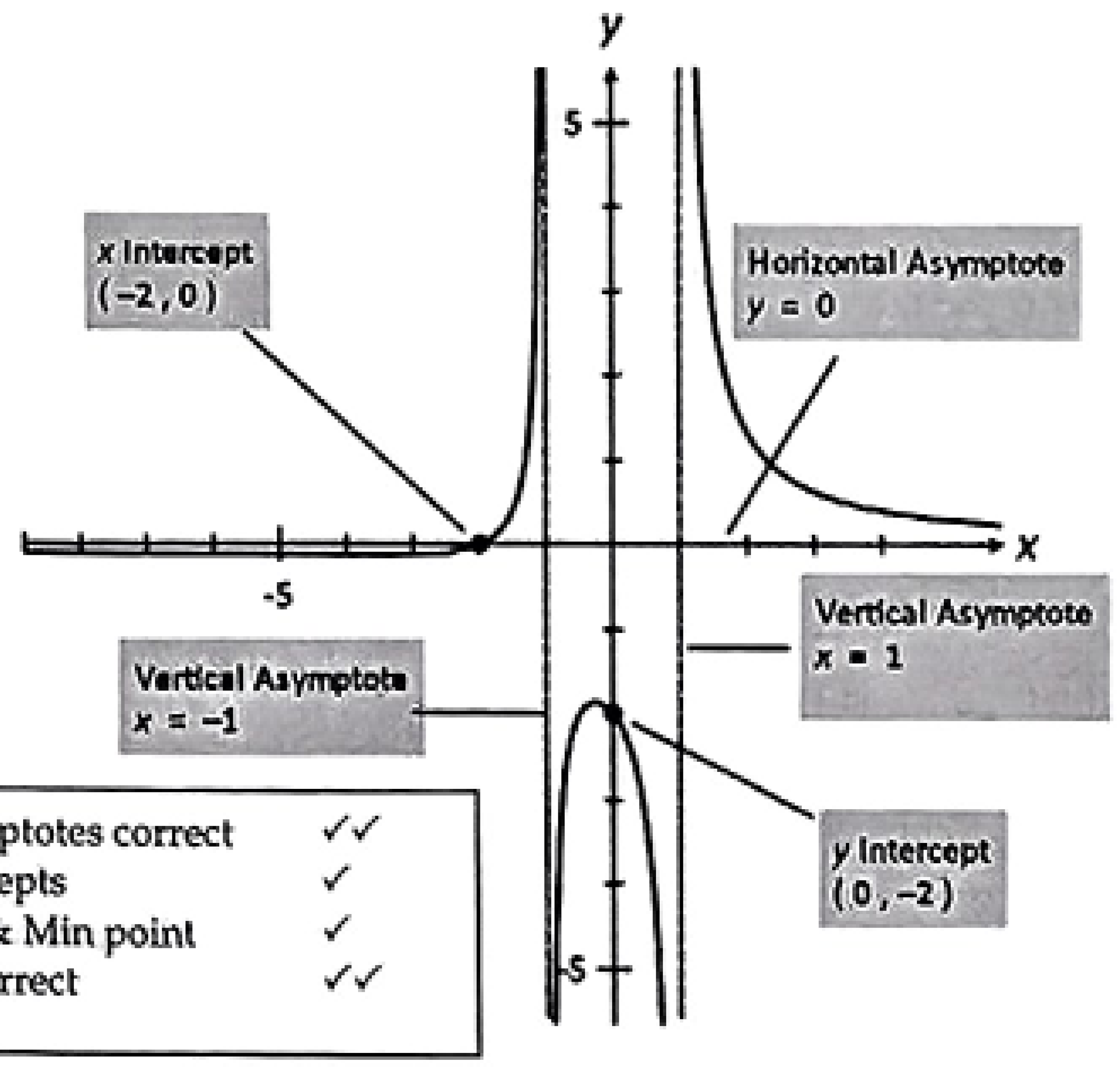
(a) State the equation of all asymptotes.

> Asymptotes: $x = -1$, $x = 1$ and $y = 0$ ✓✓

(b) Show that for $x < -2$, $y < 0$.

> For $x < -2$, $x + 2 < 0$ and $x^2 - 1 > 0$ ✓✓
> Hence, quotient $\dfrac{x+2}{x^2-1} < 0$. ✓

(c) Sketch this curve. Indicate all intercepts and asymptotes.



Asymptotes correct	✓✓
Intercepts	✓
Max & Min point	✓
All correct	✓✓

Calculator Free

8. [9 marks: 2, 3, 4]

Consider the curve with equation $y = \dfrac{x^2+x-2}{x^2-2x-8}$.

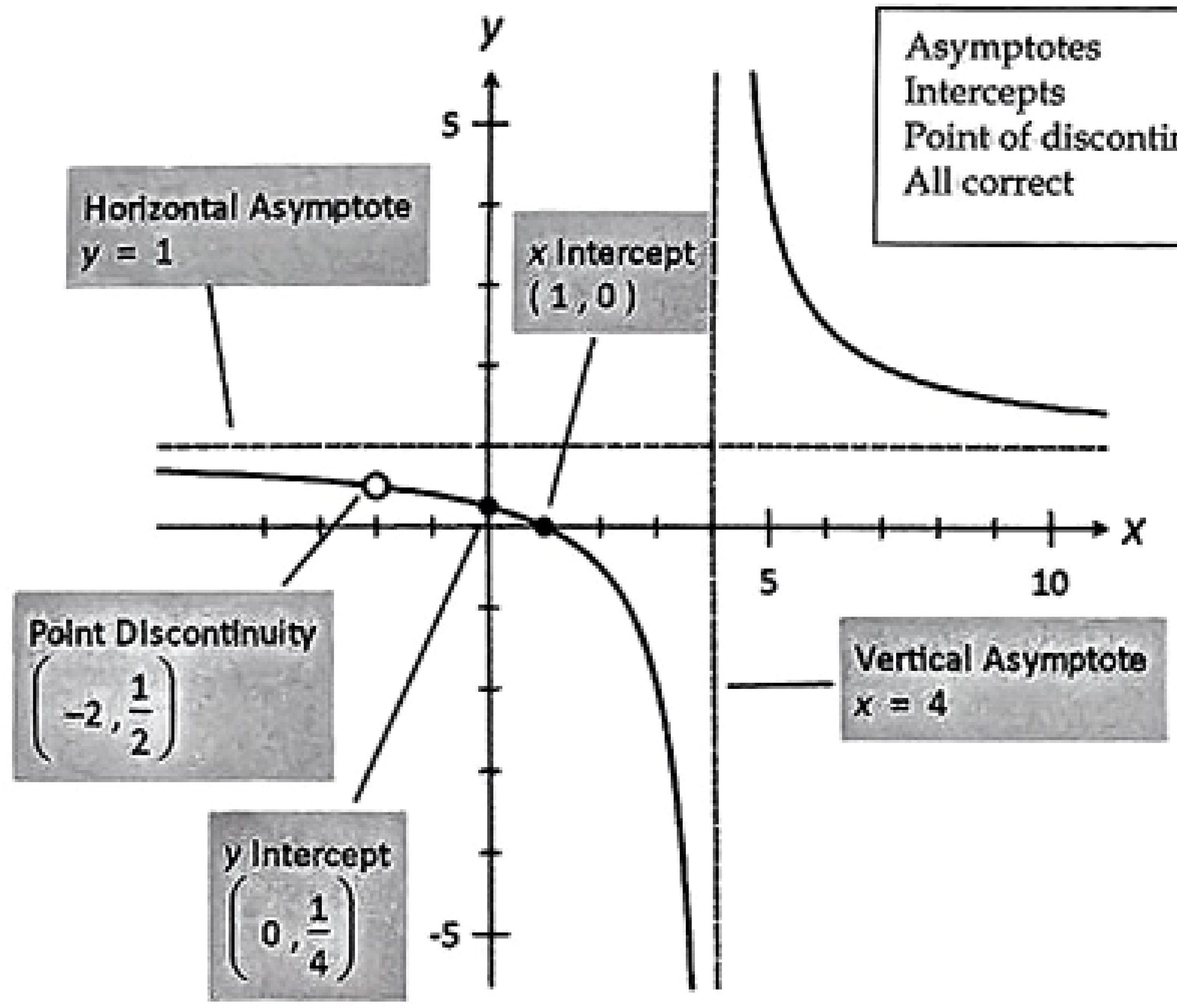
(a) State the equation of all asymptotes.

> $y = \dfrac{(x-1)(x+2)}{(x-4)(x+2)}$
> Asymptotes are: $x = 4$ and $y = 1$ ✓✓

(b) Identify the point of discontinuity on this curve.

> $y = \dfrac{(x-1)(x+2)}{(x-4)(x+2)}$
> $= \dfrac{(x-1)}{(x-4)}$ for $x \neq -2$ ✓✓
> For $x \to -2 \Rightarrow y \to -\dfrac{1}{2}$
> Hence, point of discontinuity is $(-2, -\frac{1}{2})$. ✓

(c) Sketch this curve on the axes provided below.



Asymptotes	✓
Intercepts	✓
Point of discontinuity	✓
All correct	✓

Calculator Free

9. [11 marks: 4, 2, 5]

Consider the curve with equation $y = \dfrac{x^2+x-6}{x-1}$.

(a) Rewrite the equation of the curve in the form $y \equiv \dfrac{P(x)}{Q(x)} + ax + b$ where $\dfrac{P(x)}{Q(x)}$ is a rational proper fraction and a and b are real constants.

$$\frac{x^2+x-6}{x-1} = \frac{x(x-1)+2x-6}{x-1} \quad ✓$$
$$= \frac{x(x-1)+2(x-1)-4}{x-1} \quad ✓$$
$$= \frac{-4}{x-1}+x+2 \quad ✓✓$$

OR

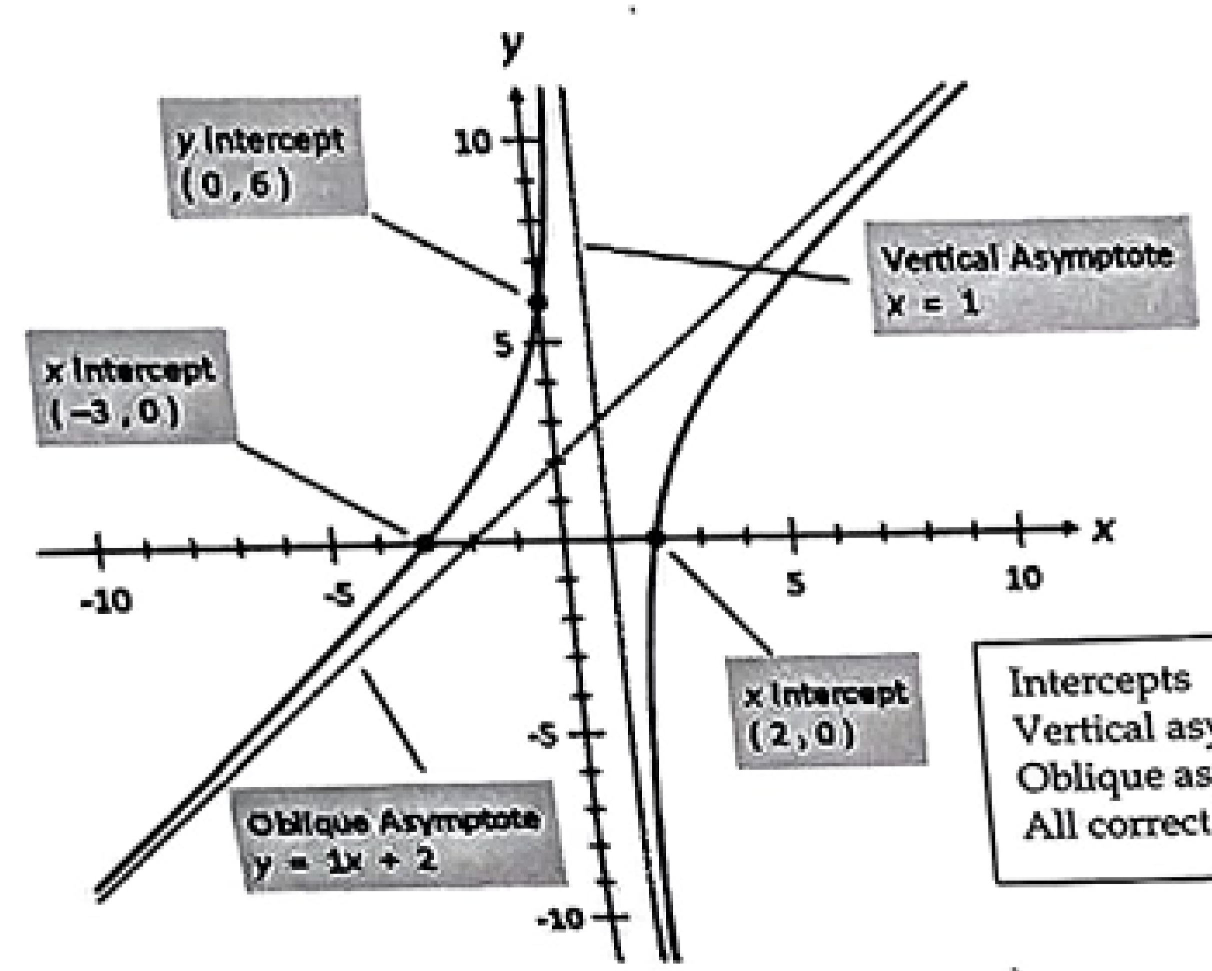
$$x-1\overline{)x^2+x-6} = x+2 \quad ✓✓$$
$$x^2-x$$
$$2x-6$$
$$2x-2$$
$$-4$$
$$y = \frac{-4}{x-1}+x+2 \quad ✓✓$$

(b) State the equations of all asymptotes of this curve.

Asymptotes: $x = 1$, ✓
$y = x+2$ ✓

(c) On the axes provided below sketch the graph of $y = \dfrac{x^2+x-6}{x-1}$.

Indicate all intercepts and asymptotes.



Intercepts ✓✓
Vertical asymptote ✓
Oblique asymptote ✓
All correct ✓

Calculator Free

10. [10 marks: 3, 4, 3]

Let $y = \dfrac{ax^3+bx+c}{x^2+k}$ where a, b, c and k are real constants.

(a) Rewrite $y = \dfrac{ax^3+bx+c}{x^2+k}$ in the form $y = \dfrac{P(x)}{Q(x)} + px + q$ where $\dfrac{P(x)}{Q(x)}$ is a rational proper fraction and p and q are real constants.

$$\frac{ax^3+bx+c}{x^2+k} = \frac{ax(x^2+k)+(b-ak)x+c}{x^2+k} \quad ✓✓$$
$$= \frac{(b-ak)x+c}{x^2+k}+ax \quad ✓$$

OR

$$x^2+k\overline{)ax^3+bx+c} = ax \quad ✓✓$$
$$ax^3+akx$$
$$(b-ak)x+c$$
$$y = \frac{(b-ak)x+c}{x^2+k}+ax \quad ✓$$

(b) The curve has intercepts only at (–2, 0) and (0, –1) and asymptotes with equation $y = x$.

(i) Determine the value of a and express b and c in terms of k.

When $x \to \infty$, $y = x$ $\Rightarrow a = 1$ ✓

When $x = 0$, $y = -1$: $\dfrac{c}{k} = -1 \Rightarrow c = -k$ ✓

When $x = -2$, $y = 0$: $-8 - 2b + c = 0$ ✓

$$b = \frac{c-8}{2}$$
$$= \frac{-k-8}{2} \quad ✓$$

(ii) Give a possible set of values for b, c and k if in addition, the curve has no singularities and no vertical asymptotes.

Necessary condition $k > 0$.

Hence, $k = t^2$, $c = -t^2$ and $b = \dfrac{-t^2-8}{2}$ ✓✓✓